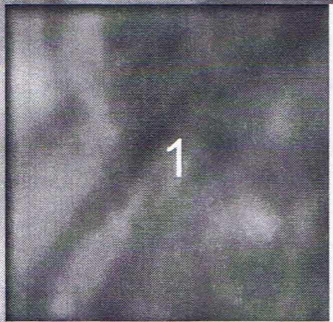


# 1 Geometry



In this chapter we are attempting to give neither a rigorous introduction to geometry nor even a summary of a traditional high school geometry course. Rather, we will try to give a brief overview that highlights basic definitions, terminology, theorems, and formulas. We pay particular attention to those results that are often encountered in algebra, precalculus, and calculus courses.

## G.1 Angles

We begin by assuming that certain undefined terms are intuitively understood. For example, we all understand what is meant by a point and by a straight line. Points and lines are often named with letters; lines are also frequently named by the points through which they pass. Thus, in Figure G.1(**a**) the line passing through the points $A$ and $B$ can be referred to as line $L$ and is usually denoted by $\overleftrightarrow{AB}$. The arrows indicate that the line goes on forever in both directions. On the other hand, the straight ***line segment*** from $A$ to $B$ is denoted by $\overline{AB}$. $A$ and $B$ are called the ***endpoints*** of the line segment. See Figure G.1(**b**).

**Figure G.1**
The length of the line segment $\overline{AB}$ is denoted as $|\overline{AB}|$. Thus in Figure G.1(**b**), $|\overline{AB}| = 5$.

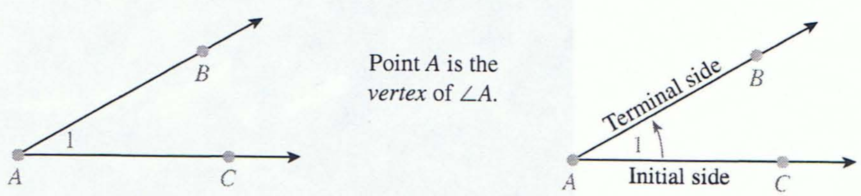

(**a**) The line $L$ or $\overleftrightarrow{AB}$ (**b**) The line segment $\overline{AB}$

We may also have occasion to talk about a ***half-line*** or ***ray***. Ray $\overrightarrow{AB}$ appears in Figure G.2. Ray $\overrightarrow{AB}$ has only one endpoint: the point $A$.

**Figure G.2**
The ray $\overrightarrow{AB}$


An ***angle*** is formed by two rays with a common endpoint. The common endpoint is called the ***vertex***. Figure G.3(**a**) shows angle $A$, which is often written $\angle A$. Note that $\angle A$ can also be referred to as $\angle BAC$, or $\angle CAB$, or $\angle 1$. When we name an angle using the three-letter designation, the middle letter must be the vertex.

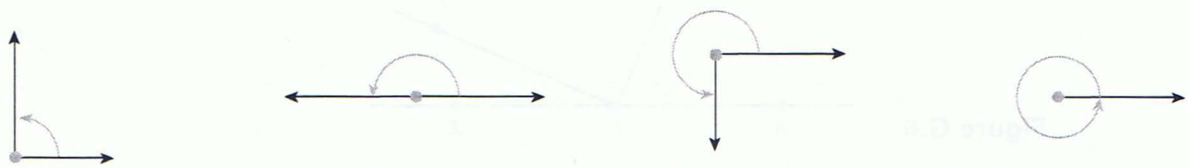
**Figure G.3**




It is helpful to think of the rays that form an angle in the following way: We keep one ray of the angle fixed (this is called the ***initial side***) and allow the second ray (this is called the ***terminal side***) to rotate to form the angle. The little arrow in Figure G.3(**b**) indicates that in $\angle A$, $\overrightarrow{AC}$ is the initial side and $\overrightarrow{AB}$ is the terminal side.

Angles are usually measured using units called ***degrees.*** If we keep in mind that when we measure an angle, we are trying to measure what part of a complete rotation we have, then we define one degree (written $1°$) to be $\frac{1}{360}$ of an entire rotation. Figure G.4(**a**) shows an angle formed by one complete rotation; Figure G.4(**b**) shows several angles and their degree measurements.

(**a**) An angle of 1 complete rotation or 360°



90° $\frac{1}{4}$ of a complete rotation | 180° $\frac{1}{2}$ of a complete rotation | 270° $\frac{3}{4}$ of a complete rotation | 360° 1 complete rotation

**Figure G.4** (**b**)

An ***acute*** angle is an angle whose measure is less than 90°.

A ***right*** angle is an angle whose measure is equal to 90°.

An ***obtuse*** angle is an angle whose measure is greater than 90° but less than 180°.

A ***straight*** angle is an angle whose measure is equal to 180°.

An example of each kind of angle is shown in Figure G.5(**a**). Note that a right angle is indicated by a little square at the vertex.

Lines that intersect at right angles are called ***perpendicular lines.***

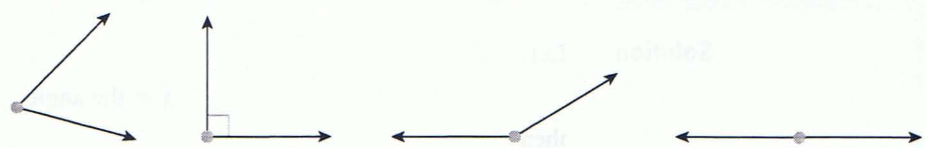


**Figure G.5(a)** Acute angle | Right angle | Obtuse angle | Straight angle

Two angles are called ***complementary*** if together they form a right angle (90°).

Two angles are called ***supplementary*** if together they form a straight angle (180°).

In Figure G.5(**b**), angles $\angle CBD$ and $\angle DBE$ are complementary, since together they form right angle $\angle CBE$, whereas angles $\angle ABD$ and $\angle DBE$ are supplementary, since together they form straight angle $\angle ABE$.

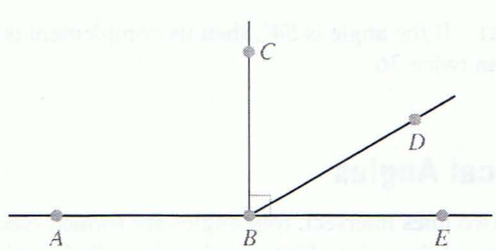



**Figure G.5(b)**

Numerically, a 30° angle and a 60° angle are complementary, since they add up to 90°, whereas a 45° angle and a 135° angle are supplementary, since they add up to 180°. (To keep the definitions of complementary and supplementary straight, you might find it helpful to remember that "c" comes before "s" and 90 comes before 180.)

**EXAMPLE 1** In Figure G.6, find $x$.

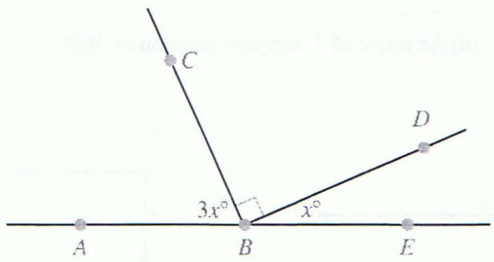



**Figure G.6**

**Solution** Since $\angle ABE$ is a straight angle, and a straight angle is 180°, we have the following:

$$\angle ABC + \angle CBD + \angle DBE = 180 \quad \textit{$\angle ABC = 3x$, $\angle DBE = x$, and since $\angle CBD$ is a right angle, $\angle CBD = 90°$, and we have}$$

$$3x + 90 + x = 180 \quad \textit{Solve for } x$$

$$4x + 90 = 180$$

$$4x = 90 \quad \textit{Divide each side by 4 to get}$$

$$x = \frac{90}{4}$$

$$x = \frac{45}{2} = \boxed{22.5}$$

**EXAMPLE 2** An angle is 18° less than twice its complement. Find the angle.

**Solution** Let

$$x = \text{the angle}$$

then

$$90 - x = \text{the complement of the angle} \quad \textit{Since their sum is } 90°$$

The given information is translated as

$$x = 2(90 - x) - 18$$
$$x = 180 - 2x - 18$$
$$x = 162 - 2x$$
$$3x = 162$$
$$x = 54°$$

**Check:** If the angle is 54°, then its complement is $90° - 54° = 36°$. Also 54° is 18° less than twice 36°.

## Vertical Angles

When two lines intersect, four angles are formed (see Figure G.7). The angles on opposite sides of the point of intersection are called *vertical angles.*

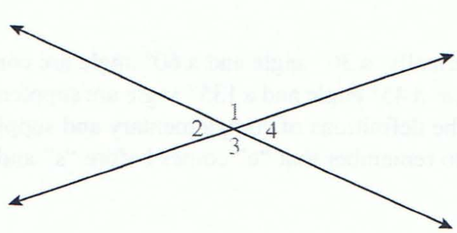



**Figure G.7**
$\angle 1$ and $\angle 3$ are vertical angles, as are $\angle 2$ and $\angle 4$.

From Figure G.6 we can see that $\angle 1 + \angle 2 = 180°$ and $\angle 2 + \angle 3 = 180°$. Therefore, it follows that $\angle 1 + \angle 2 = \angle 2 + \angle 3$, and hence, $\angle 1$ must be equal to $\angle 3$. We have just demonstrated the truth of the following theorem.

**THEOREM G.1** Vertical angles are equal.

## EXAMPLE 3

In Figure G.8, lines $L$ and $R$ intersect as indicated. Find the measures of the remaining angles.

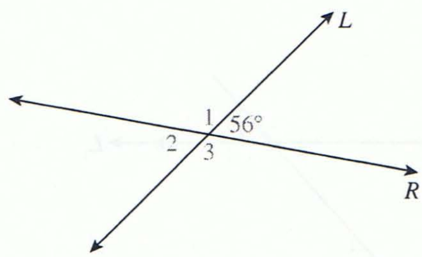



**Figure G.8**

**Solution** Since $\angle 2$ and the angle of 56° are vertical angles and vertical angles are equal, we have $\boxed{\angle 2 = 56°}$.

We can see that $\angle 1$ and the 56° angle form a straight angle. Therefore, $\angle 1$ is supplementary to a 56° angle, so $\angle 1 = 124°$. Since $\angle 1$ and $\angle 3$ are vertical angles, we have $\boxed{\angle 1 = \angle 3 = 124°}$.

## EXAMPLE 4

In Figure G. 9, find the measure of each angle.

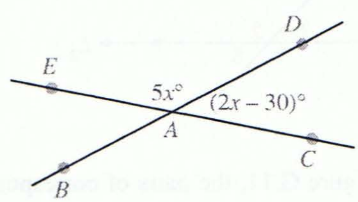



**Figure G.9**

**Solution** Since $\angle EAC$ is a straight angle and a straight angle is 180°, we have the following:

$$\angle EAD + \angle DAC = 180 \quad \textit{Since } \angle EAD = 5x \textit{ and } \angle DAC = 2x - 30, \textit{ we have}$$
$$5x + 2x - 30 = 180 \quad \textit{Solve for } x.$$
$$7x - 30 = 180$$
$$7x = 210$$
$$x = 30$$

Since $x = 30$, we have

$$\angle EAD = 5x = 5(30) = 150°$$
$$\angle DAC = 2x - 30 = 2(30) - 30 = 30°$$
$$\angle BAC = 150°, \quad \text{since } \angle BAC \text{ and } \angle EAD \text{ are vertical angles}$$
$$\angle EAB = 30°, \quad \text{since } \angle EAB \text{ and } \angle DAC \text{ are vertical angles}$$

## Parallel Lines

In everyday usage we may describe parallel lines as "lines that never meet." However, this is not a practical definition, since it is hard to check whether two given lines ever meet. Instead, given two lines, we call any line (or line segment) that crosses these two lines a ***transversal.*** We can then say that two lines are parallel if they are going in the same direction with respect to any transversal, which means that they make the same angles with the transversal.

In Figure G.10 we see lines $L_1$ and $L_2$ being crossed by the transversal $M$. We can see that requiring the lines to go in the same direction requires that $\angle 1 = \angle 5$ or $\angle 3 = \angle 5$. The arrows on the lines indicate that line $L_1$ is parallel to $L_2$. Angles 1 and 5 are called ***corresponding angles.*** Angles 3 and 5 are called ***alternate interior angles.***

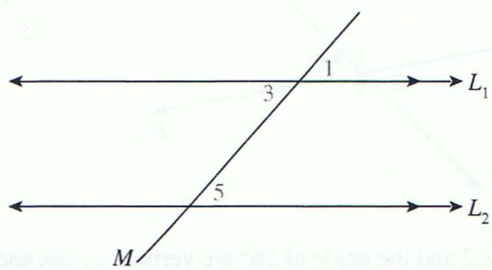



**Figure G.10**

In general, when two lines are crossed by a transversal, four pairs of corresponding angles are formed and two pairs of alternate interior angles are formed. See Figure G.11.

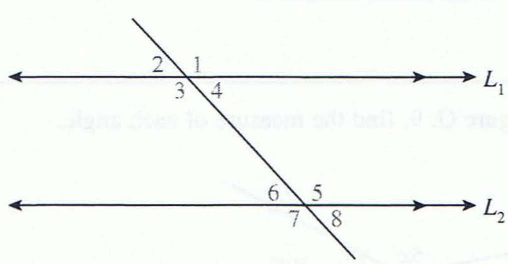



**Figure G.11**

Referring to Figure G.11, the pairs of corresponding angles and alternate interior angles are as follows:

| **Pairs of corresponding angles** | **Pairs of alternate interior angles** |
|---|---|
| $\angle 1$ and $\angle 5$ | $\angle 3$ and $\angle 5$ |
| $\angle 2$ and $\angle 6$ | $\angle 4$ and $\angle 6$ |
| $\angle 3$ and $\angle 7$ | |
| $\angle 4$ and $\angle 8$ | |

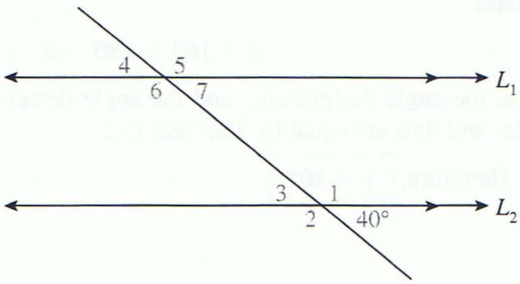
We now state the following theorem.

**THEOREM G.2** In the situation described by Figure G.11, lines $L_1$ and $L_2$ are parallel if and only if *any* pair of corresponding angles or alternate interior angles is equal.

## EXAMPLE 5

Fill in all the missing angles in the following diagram.

*Remember that the arrows on the lines tell us that the lines are parallel.*




**Solution** Since $\angle 3$ is a vertical angle to the angle of 40°, $\boxed{\angle 3 = 40°}$.

Since $\angle 1$ is supplementary to 40° (together they form a straight angle), we have $\boxed{\angle 1 = 140°}$.

Similarly, since $\angle 2$ is vertical to $\angle 1$, $\boxed{\angle 2 = 140°}$.

Since the two lines $L_1$ and $L_2$ are parallel, the corresponding angles are equal.

Therefore, we have $\angle 1 = \angle 5$, and so $\boxed{\angle 5 = 140°}$.

Finally, since $\angle 6$ corresponds to $\angle 2$, and $\angle 4$ corresponds to $\angle 3$ and is vertical to $\angle 7$, $\boxed{\angle 6 = 140°, \angle 4 = \angle 7 = 40°}$.

## EXAMPLE 6

In Figure G.12, find the measure of $y$.

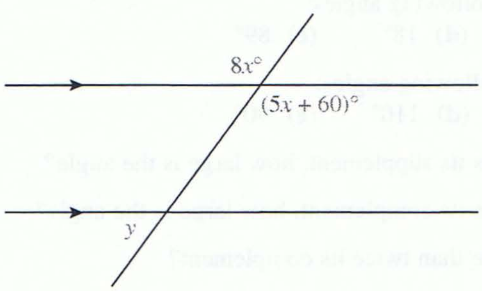



**Figure G.12**

**Solution** The angles labeled $8x$ and $5x + 60$ are vertical angles and therefore are equal, so we have

$$8x = 5x + 60$$
$$3x = 60$$
$$x = 20$$

Since $x = 20$, the angle designated $8x$ is $8(20) = 160°$. Note the angle designated $5x + 60$ is also 160°, since it is vertical to the angle designated $8x$.

Let's call $z$ the angle corresponding to the angle designated $y$ (see Figure G.13).

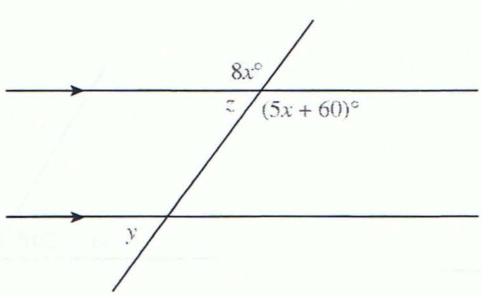



**Figure G.13**

The angle designated $z$ is supplementary to the angle designated $8x$ [and to the angle designated $(5x + 60)$], and since the angle designated $8x$ has a measure of 160°, we have

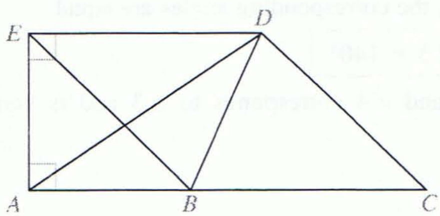
$$z + 160 = 180 \quad \Rightarrow \quad z = 20^\circ$$

Since, the angle designated $z$ and the angle designated $y$ are a pair of corresponding angles and thus are equal by Theorem G.2.

Therefore, $\boxed{y = 20^\circ}$.

## EXERCISES G.1

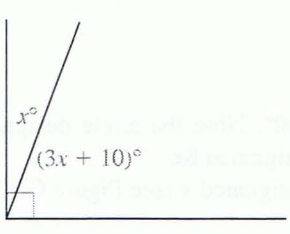
1. Given the following figure, which pairs of angles are complementary?



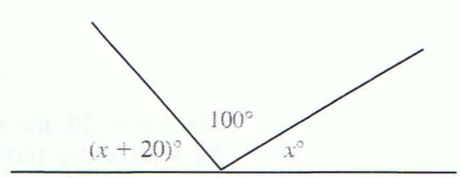

2. Given the same figure as in Exercise 1, which pairs of angles are supplementary?
3. Find the complement of each of the following angles:
**(a)** 30° **(b)** 60° **(c)** 45° **(d)** 18° **(e)** 89°
4. Find the supplement of each of the following angles:
**(a)** 15° **(b)** 80° **(c)** 24° **(d)** 110° **(e)** 90°
5. If an angle is 28° less than three times its supplement, how large is the angle?
6. If an angle is 28° less than three times its complement, how large is the angle?
7. How large is an angle if it is 12° more than twice its complement?
8. How large is an angle if it is 12° more than twice its supplement?

*In Exercises 9–12, find x.*

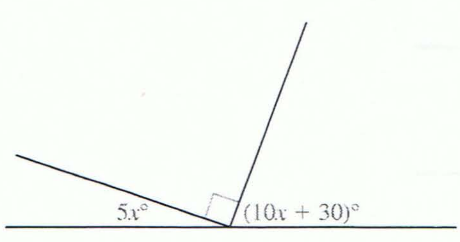
**9.**



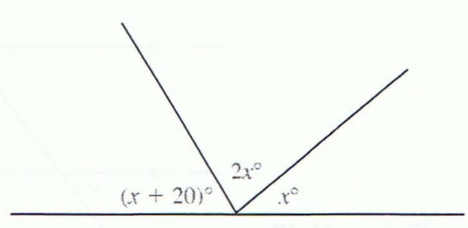

**10.**



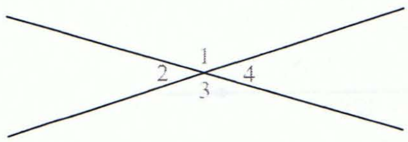

**11.**



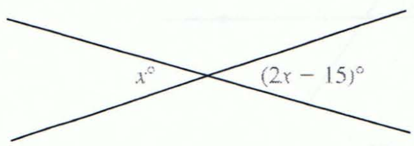

**12.**



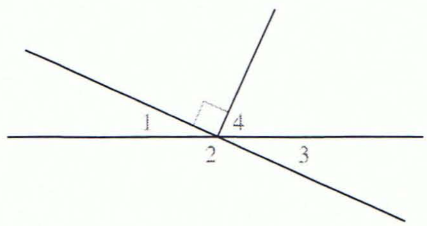
**13.** In the following figure, $\angle 4 = 35°$. Find the remaining angles.



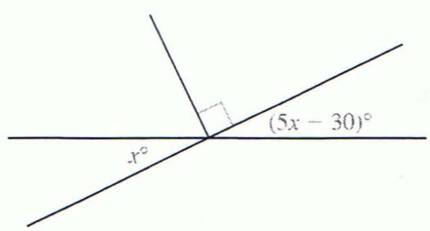

**14.** Find $x$.



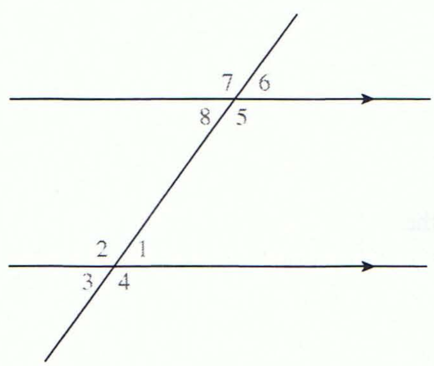

**15.** In the following figure $\angle 4 = 25°$. Find the measures of angles 1, 2, and 3.



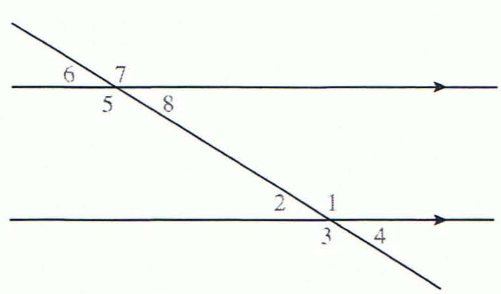

**16.** Find $x$.



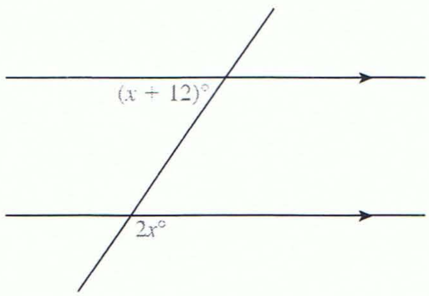

**17.** In the following figure, $\angle 4 = 125°$. Find all the remaining angles.



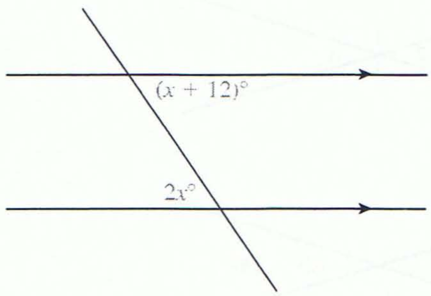

**18.** In the following figure, $\angle 8 = 32°$. Find all the remaining angles.



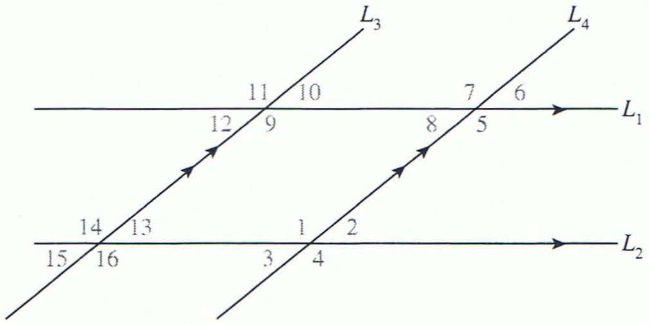
**19.** Find $x$.



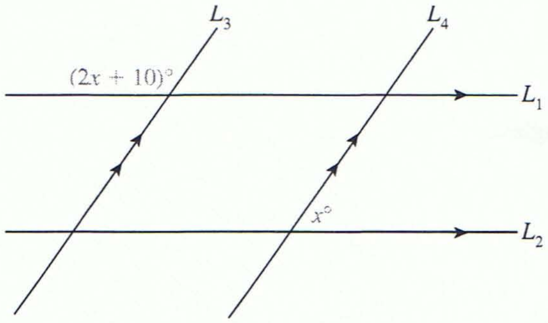

**20.** Find $x$.



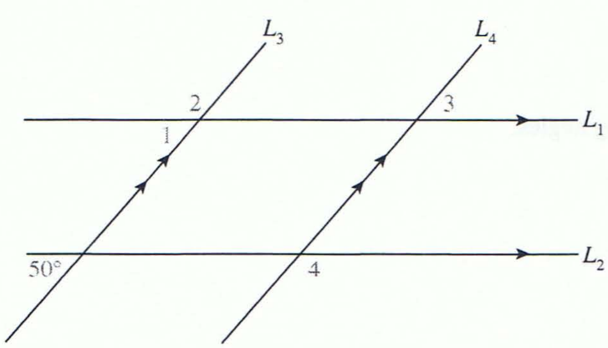

**21.** In the following figure, $L_1$ is parallel to $L_2$, $L_3$ is parallel to $L_4$, and $\angle 2 = 40°$. Find all the remaining angles.



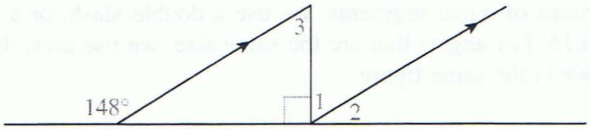

**22.** In the following figure, $L_1$ is parallel to $L_2$ and $L_3$ is parallel to $L_4$. Find $x$.




**23.** In the following figure, $L_1$ is parallel to $L_2$ and $L_3$ is parallel to $L_4$. Find the measures of angles 1, 2, 3, and 4.



**24.** Find the measures of angles 1, 2, and 3.




## G.2 Triangles

When a portion of the plane is totally enclosed by straight line segments, the enclosing figure is called a ***polygon.*** Polygons are usually named according to the number of sides that they have. A polygon of three sides is called a ***triangle;*** of four sides is called a ***quadrilateral;*** of five sides is called a ***pentagon;*** etc. In this section we will concentrate on triangles.

Triangles are often categorized by the number of sides of equal length.

An ***equilateral*** triangle has three equal sides.

An ***isosceles*** triangle has two equal sides.

A ***scalene*** triangle has all three sides unequal.

One perhaps somewhat surprising fact is that every triangle, regardless of its size or shape, has the property that the sum of its three angles is 180°.

We can prove this fact by considering any triangle $ABC$ (often denoted as $\triangle ABC$) with line $L$ drawn through $C$ and parallel to $\overline{AB}$, as in Figure G.14.

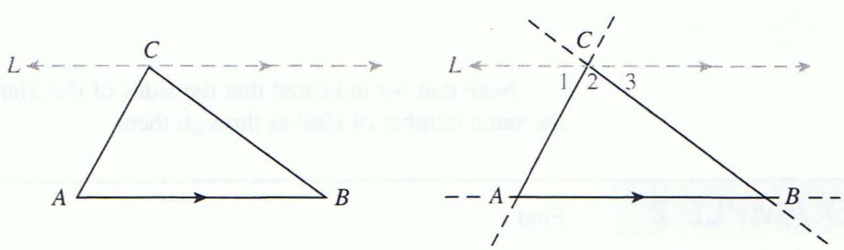



**Figure G.14**

We can see that $\angle 1 = \angle A$ and $\angle 3 = \angle B$ because they are alternate interior angles. $\angle 2$ is just another name for $\angle C$ of the triangle.

Thus, we see that $\angle A + \angle B + \angle C = \angle 1 + \angle 2 + \angle 3 = 180°$, and we can state the following theorem.

**THEOREM G.3** The sum of the angles of a triangle is 180°.

Triangles have the property that within a specific triangle, the longer the side, the larger the opposite angle and vice versa. It then follows that if any of the sides of the triangle are of equal length, the angles opposite those sides will also be equal. Thus a triangle is equilateral (all three sides are equal), if and only if all three angles are equal; the triangle is isosceles (two sides are equal) if and only if two angles are equal.

The angles opposite the equal sides of an isosceles triangle are called the *base angles* of the isosceles triangle.

The base angles of an isosceles triangle are equal.

*A word about notation:* In figures, when we want to indicate that two line segments are the same size, we put a small slash through each of the segments that are equal. If there are other groups of equal segments, we use a double slash, or a triple slash as shown in Figure G.15. For angles that are the same size, we use arcs, double arcs, and triple arcs as shown in the same figure.

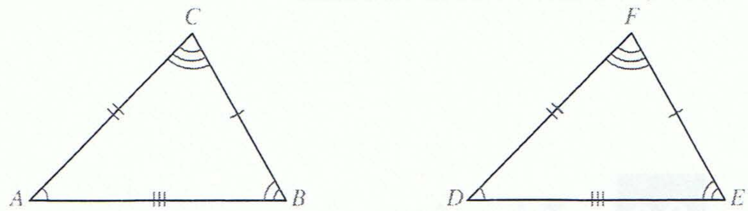
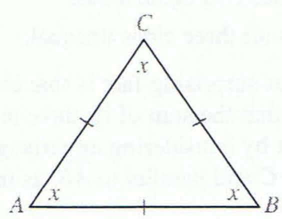



**Figure G.15**

**EXAMPLE 1** Find the number of degrees in each angle of an equilateral triangle.

**Solution** From the comments we have just made, the fact that the triangle is equilateral means that all the sides are equal and so all the angles must be equal. Therefore, we have labeled all three angles with the same letter, $x$. Since the angles must all be equal, and the sum of the angles is 180°, we can write

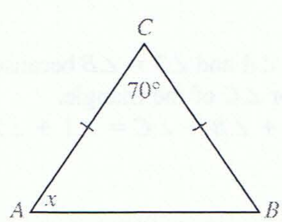



$$3x = 180°$$

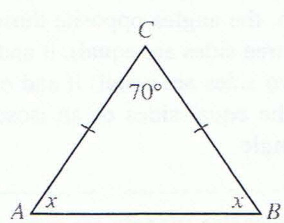
$$\boxed{x = 60°}$$

Note that we indicated that the sides of the triangle are of equal length by putting the same number of slashes through them.

**EXAMPLE 2** Find $x$.

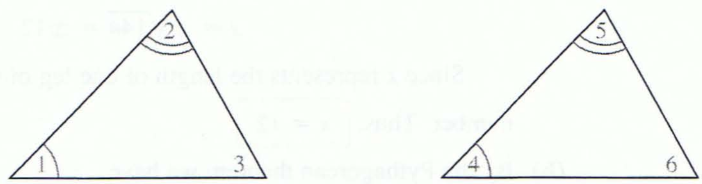



**Solution** Since the triangle is isosceles (because the diagram tells us that $|\overline{AC}| = |\overline{BC}|$), $\angle A$ must be equal to $\angle B$, since they are base angles. Thus we label both $\angle A$ and $\angle B$ as $x$ in the figure below, and write the following equation:




$$x + x + 70° = 180°$$
$$2x + 70° = 180°$$
$$2x = 110°$$
$$\boxed{x = 55°}$$

Suppose two angles of one triangle are equal to two angles of another triangle, such as the example given in the figure below: $\angle 1 = \angle 4$ and $\angle 2 = \angle 5$. Since the sum of the three angles in any triangle is 180°, we can clearly see that the remaining two angles, $\angle 3$ and $\angle 6$, are equal to each other.

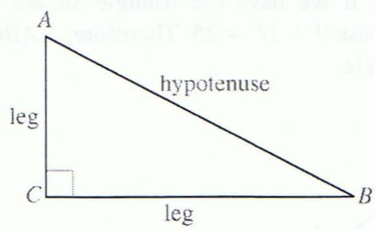



Thus we have the following:

> If two angles in one triangle are equal to two angles in another triangle, then the third angles in each triangle are equal to each other.

If one of the angles of a triangle is a right angle, then the triangle is called a ***right triangle.*** The sides that form the right angle are called the ***legs,*** and the side opposite the right angle is called the ***hypotenuse.*** Figure G.16 shows right triangle $\triangle ABC$. The legs are $\overline{AC}$ and $\overline{BC}$, and the hypotenuse is $\overline{AB}$.

**Figure G.16**

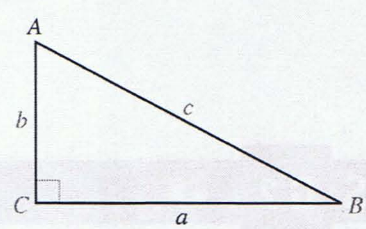



## Pythagorean Theorem

One of the most famous theorems in all of mathematics is named for the Greek mathematician Pythagoras.

**THEOREM G.4 THE PYTHAGOREAN THEOREM**

In right $\triangle ABC$, $a^2 + b^2 = c^2$.

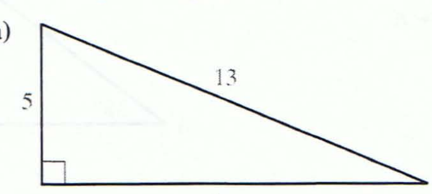



In words, the Pythagorean theorem says that "the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse."

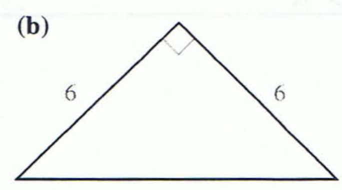
**EXAMPLE 3**

For each right triangle, find $x$.

**(a)**




**(b)**



**Solution** (a) By the Pythagorean theorem we have

$$x^2 + 5^2 = 13^2$$
$$x^2 + 25 = 169$$
$$x^2 = 144 \qquad \textit{Take square roots.}$$
$$x = \pm\sqrt{144} = \pm 12$$

Since $x$ represents the length of one leg of the triangle, $x$ must be a positive number. Thus, $\boxed{x = 12}$.

(b) By the Pythagorean theorem we have

$$x^2 = 6^2 + 6^2$$
$$x^2 = 72 \qquad \textit{Take square roots.}$$
$$x = \pm\sqrt{72} \qquad \textit{Simplify the radical to get}$$
$$x = \pm\sqrt{36 \cdot 2} = \pm\sqrt{36}\,\sqrt{2} = \pm 6\sqrt{2}$$

Since $x$ represents a length and must be a positive $\boxed{x = 6\sqrt{2}}$.

We also point out that the converse of the Pythagorean theorem is true. That is, if we have a triangle in which $a^2 + b^2 = c^2$, then the triangle must be a right triangle.

For example, if we have the triangle shown in Figure G.17, we can see that $3^2 + 4^2 = 5^2$ because $9 + 16 = 25$. Therefore, $\triangle ABC$ must be a right triangle with $\angle C$ being the right angle.

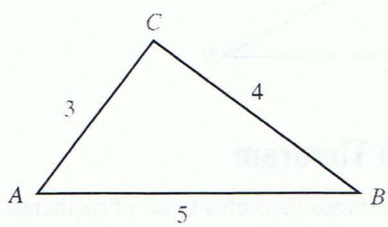
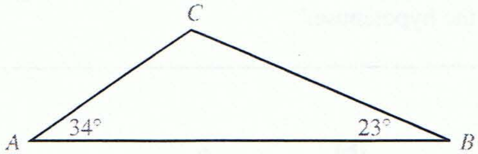



**Figure G.17**

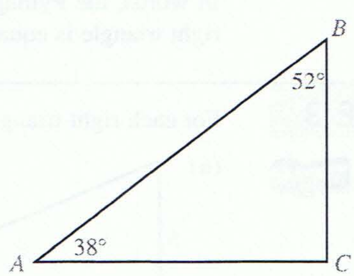
## EXERCISES G.2

*In Exercises 1–4, find all the missing angles.*

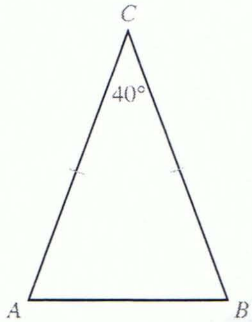
**1.**



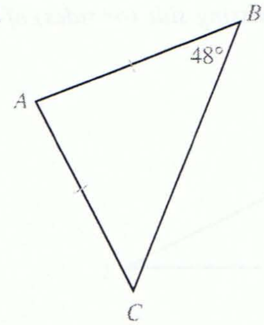

**2.**



**3.**

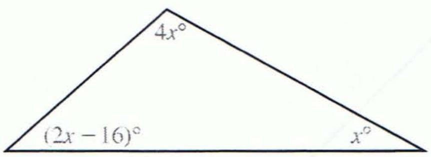



**4.**

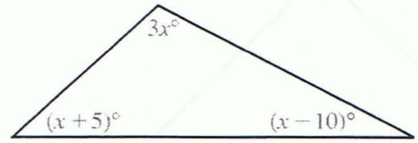



*In Exercises* 5–12, *find x.*

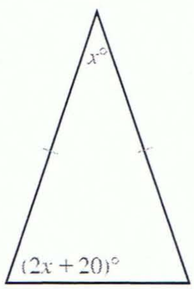
**5.**



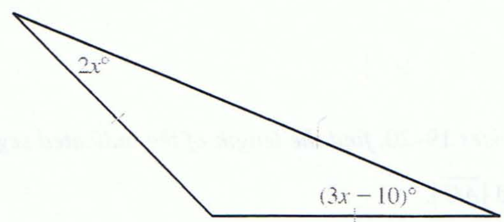

**6.**



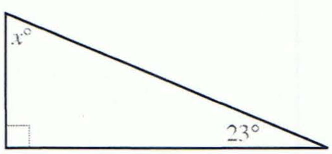

**7.**



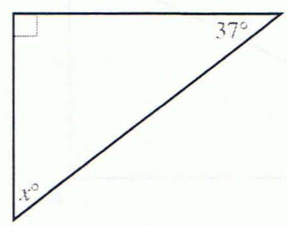

**8.**



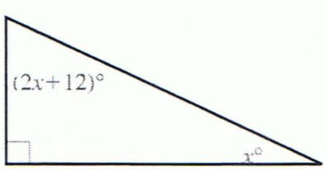

**9.**



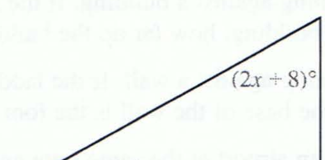

**10.**




**11.**

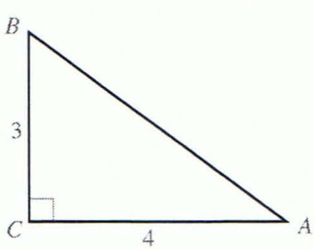



**12.**

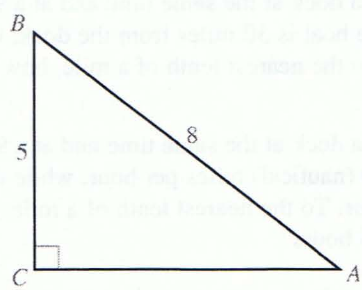



*In Exercises* 13–14, *find the missing side of each triangle.*

**13.**

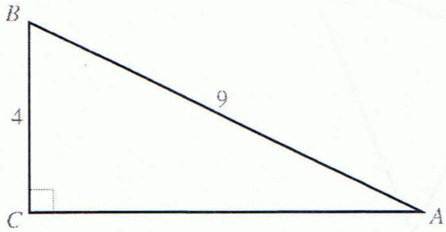



**14.**

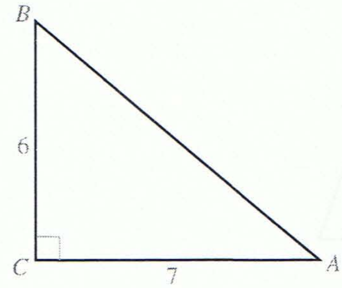


*In Exercises* 15–18, *find the missing side (or sides) of each triangle(s).*

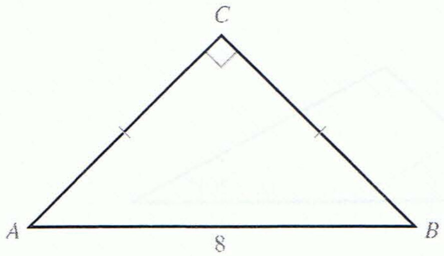
**15.**



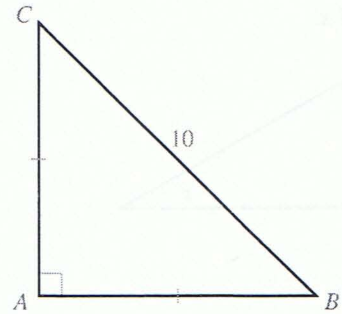

**16.**




**17.**

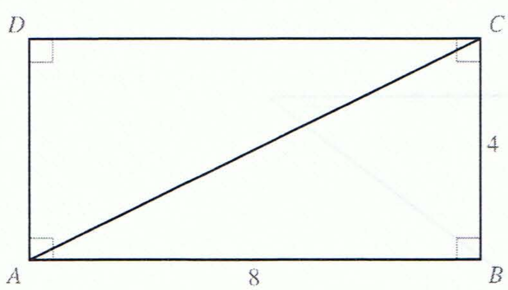



**18.**

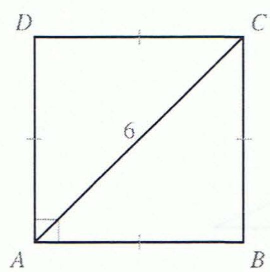



*In Exercises* 19–20, *find the length of the indicated segment.*

**19.** Find $|\overline{AC}|$.




**20.** Find $|\overline{CB}|$.




**21.** A 30-ft ladder is leaning against a building. If the foot of the ladder is 10 ft away from the base of the building, how far up the building does the ladder reach?

**22.** A 40-ft ladder is leaning against a wall. If the ladder reaches 20 ft up the wall, how far away from the base of the wall is the foot of the ladder?

**23.** Two airplanes leave an airport at the same time and at a 90° angle from each other. After an hour of flying at the same altitude, one plane is 160 miles from the airport, and the other is 180 miles from the airport. To the nearest tenth of a mile, how far are the planes from each other?

**24.** Two boats leave a dock at the same time and at a 90° angle from each other. After 3 hours one boat is 30 miles from the dock, while the other is 50 miles from the dock. To the nearest tenth of a mile, how far are the boats from each other?

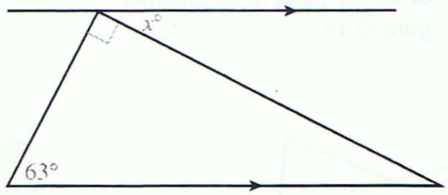
**25.** Two boats leave a dock at the same time and at a 90° angle from each other. One boat travels at 20 (nautical) miles per hour, while the other travels at 32 (nautical) miles per hour. To the nearest tenth of a mile, how far are the boats from each other after 3 hours?

**26.** Two airplanes leave an airport at the same time and at a 90° angle from each other. Both planes fly at the same altitude, one at 120 mph and the other at 140 mph. To the nearest tenth of a mile, how far are the planes from each other after 3 hours?

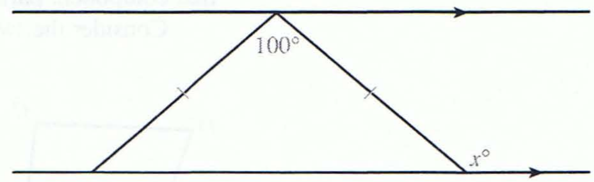
**27.** Will a 15-in. ruler fit in an $8\frac{1}{2}$- by 11-in. envelope? Explain your answer. (Assume that the width of the ruler is negligible.)

**28.** Will a 13-in. ruler fit in an $8\frac{1}{2}$- by 11-in. envelope? Explain your answer. (Assume that the width of the ruler is negligible.)

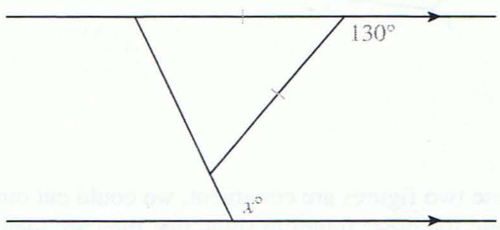
**29.** Find *x*.



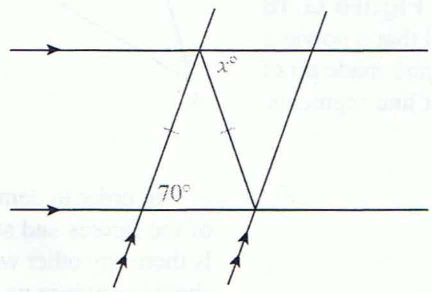

**30.** Find *x*.



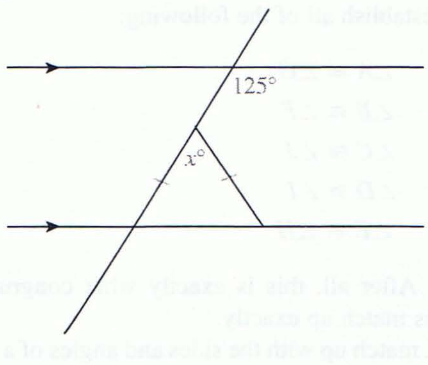

**31.** Find *x*.



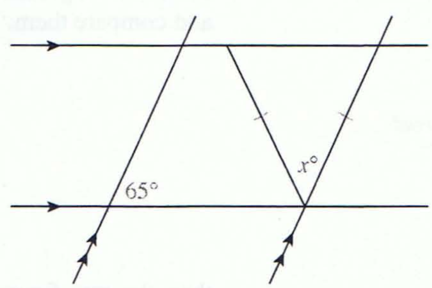

**32.** Find *x*.



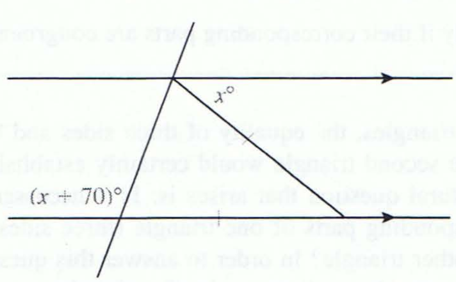

**33.** Find *x*.



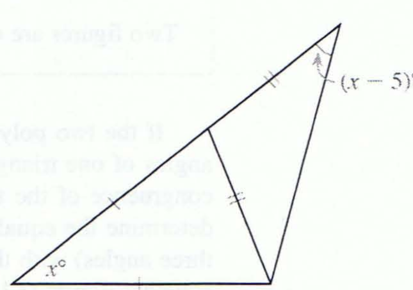

**34.** Find *x*.




**35.** Find *x*.




**36.** Find *x*.



# G.3 Congruence

When two geometric figures are *identical* (like two identical pieces of a puzzle that can be placed exactly on top of each other), the figures are called ***congruent.*** Congruent figures are exact duplicates of each other.

There are many everyday situations in which we are interested in making congruent (identical) copies of objects. For example, tracing over an existing figure creates a congruent copy; using a dress pattern to cut out pieces of fabric creates congruent pieces of material; mass producing automobile or computer parts depends on the idea that component parts must be congruent for all the parts to fit together.

Consider the two figures shown in Figure G.18

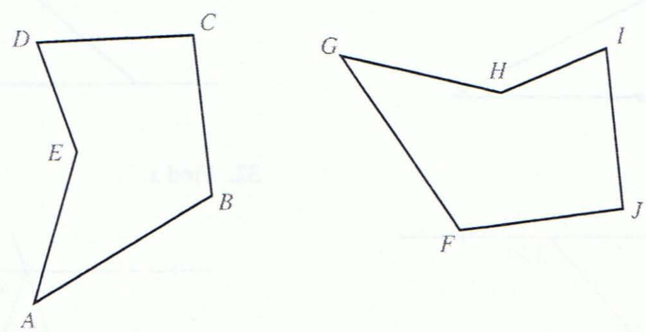



**Figure G.18**
Recall that a polygon is a figure made up of straight line segments.

In order to demonstrate that these two figures are congruent, we could cut out one of the figures and superimpose it onto the other figure to show that they are identical. Is there any other way to establish the congruence of two figures? A moment's thought should convince us that two polygons are congruent if and only if the sides and angles of one are individually congruent to the sides and angles of the other. Referring back to Figure G.18, instead of cutting out one figure and superimposing it on the other to establish congruence, we could simply measure the sides and angles of the two figures and compare them. In other words, if we establish all of the following:

*The symbol ≅ is read "is congruent to."*

$$\overline{AB} \cong \overline{GF} \qquad \angle A \cong \angle G$$
$$\overline{BC} \cong \overline{FJ} \qquad \angle B \cong \angle F$$
$$\overline{CD} \cong \overline{JI} \qquad \angle C \cong \angle J$$
$$\overline{DE} \cong \overline{IH} \qquad \angle D \cong \angle I$$
$$\overline{EA} \cong \overline{HG} \qquad \angle E \cong \angle H$$

then the two figures must be congruent. After all, this is exactly what congruence means—that all the parts of the two figures match up exactly.

The sides and angles of one figure that match up with the sides and angles of a congruent figure are called the ***corresponding parts*** of the congruent figures.

| |
|---|
| Two figures are congruent if and only if their corresponding parts are congruent. |

If the two polygons happen to be triangles, the equality of three sides and three angles of one triangle with those of the second triangle would certainly establish the congruence of the two triangles. A natural question that arises is: Is it necessary to determine the equality of all six corresponding parts of one triangle (three sides and three angles) with the six parts of the other triangle? In order to answer this question, consider Figure G.19, which illustrates two sides and one angle of a triangle.

*How many different triangles can be made given sides $\overline{AB}$, $\overline{AC}$, and $\angle A$?*

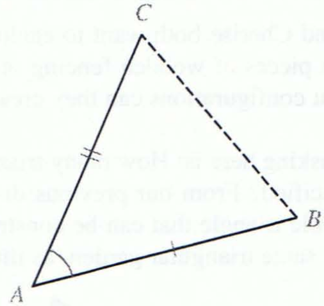



**Figure G.19**
Specifying two sides and the included angle of a triangle

How many triangles can be formed with the two specified sides and the angle (usually called the included angle) between them? The answer is clearly "one!" Our only choice is to connect points $B$ and $C$ to complete the triangle, as indicated by the dashed line in Figure G.19. In other words, if two triangles have two sides and the included angle of one equal to two sides and the included angle of another, then the triangles must be congruent. This congruence property of triangles is generally referred to as SAS (side-angle-side) and stands for the following statement:

> Two triangles are congruent if two sides and the included angle of one triangle are conguent to two sides and the included angle of the other.

In other words, we are saying: If we know that three particular parts of the two triangles are identical, then all six parts of the two triangles must be identical.

Figure G.20 illustrates two additional situations in which three parts of a triangle are sufficient to completely determine the triangle.

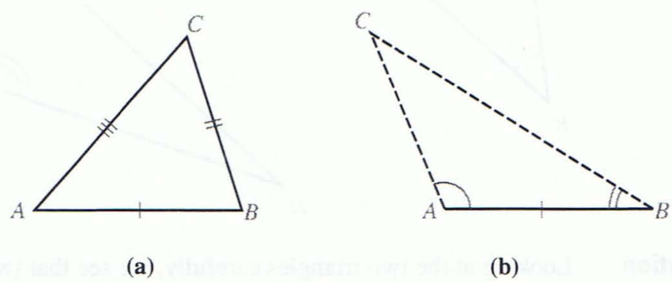



**Figure G.20**

**(a)** Figure G.20(**a**) illustrates that when all three sides of a triangle are specified, only one triangle is possible. This congruence property of triangles is generally referred to as SSS (side-side-side) and stands for the following statement:

> Two triangles are congruent if three sides of one triangle are congruent to three sides of the other.

**(b)** Figure G.20(**b**) illustrates that when two angles and the included side of a triangle are specified, only one triangle is possible. This congruence property of triangles is generally referred to as ASA (angle-side-angle) and stands for the following statement:

> Two triangles are congruent if two angles and the included side of one triangle are congruent to two angles and the included side of the other.

## EXAMPLE 1

Suppose that Lamar and Cherise both want to enclose a triangular garden. They both purchase three straight pieces of wooden fencing of lengths 5 ft, 8 ft, and 10 ft. How many possible different configurations can they create for the triangular enclosure?

**Solution** What we are actually asking here is: How many triangles can we form if the lengths of the three sides are specified? From our previous discussion, we know from SSS that there is only one possible triangle that can be constructed. Therefore, both Lamar and Cherise must build the same triangular garden, as illustrated in Figure G.21.

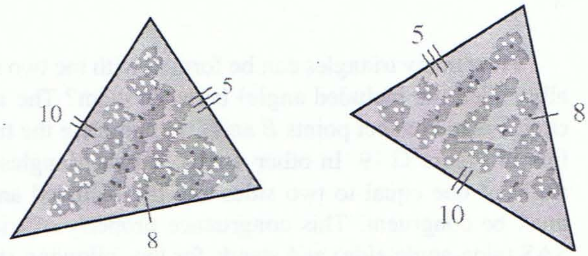
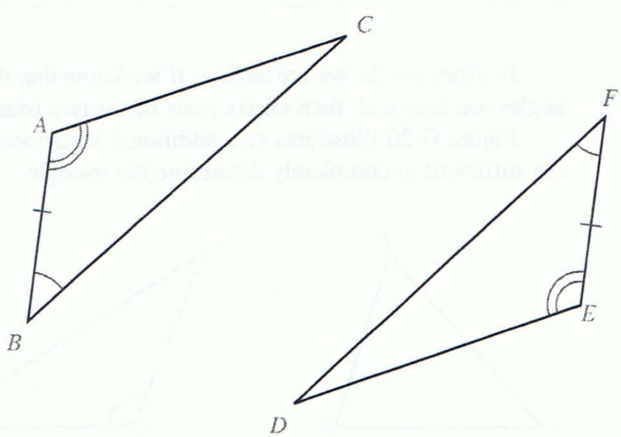



**Figure G.21** The enclosed triangular gardens of Example 1.

## EXAMPLE 2

Establish the congruence of the following triangles and determine which other parts of the two triangles are congruent




**Solution** Looking at the two triangles carefully, we see that two angles and the included side of one triangle are congruent to two angles and the included side of the second triangle. Therefore, the two triangles are congruent by ASA.

Because $\angle C$ and $\angle D$ are corresponding parts of congruent triangles, $\angle C \cong \angle D$. $\overline{AC} \cong \overline{ED}$ because these sides are opposite equal angles. Similarly, we have $\overline{BC} \cong \overline{FD}$ because these sides are opposite equal angles.

## EXAMPLE 3

In Figure G.22, $\overline{AB} \parallel \overline{DC}$ and $\overline{BC} \parallel \overline{AD}$. Explain why $\triangle ADC \cong \triangle CBA$.

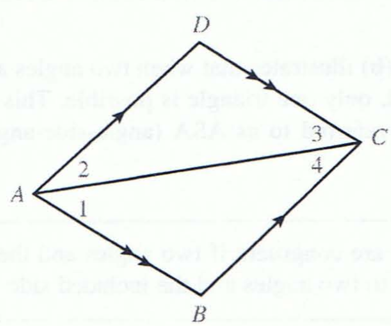



**Figure G.22**

**Solution** We can view $\overline{AC}$ as a transversal cutting across the parallel line segments $\overline{AB}$ and $\overline{DC}$. Hence, $\angle 1$ and $\angle 3$ are alternate interior angles to this pair of parallel lines and are therefore congruent: $\angle 1 \cong \angle 3$ (see Figure G.23(**a**)). On the other hand, we can view $\overline{AC}$ as a transversal cutting across the parallel line segments $\overline{AD}$ and $\overline{BC}$. Hence, $\angle 2$ and $\angle 4$ are alternate interior angles to this pair of parallel lines, and therefore congruent: $\angle 2 \cong \angle 4$ (see Figure G.23(**b**)). Both triangles $\triangle ABC$ and $\triangle CDA$ share the common side $\overline{AC}$. By ASA, $\triangle ADC \cong \triangle CBA$ (see Figure G.23(**c**)).

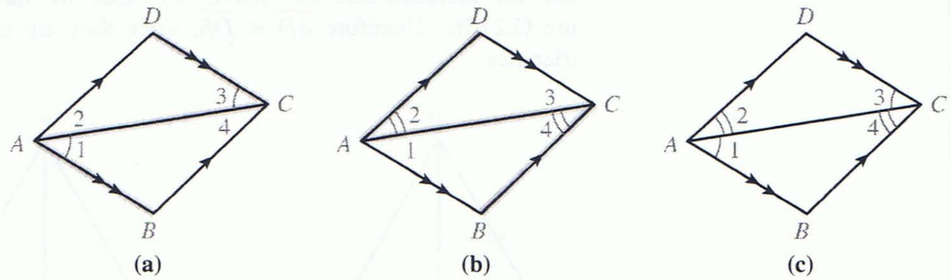



**Figure G.23**

## EXAMPLE 4

In Figure G.24, explain why $\overline{ED} \cong \overline{BC}$.

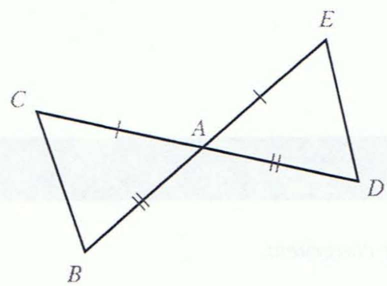



**Figure G.24**

**Solution** We are given that $\overline{AE} \cong \overline{AC}$ and $\overline{AD} \cong \overline{AB}$. Since $\angle CAB$ and $\angle EAD$ are vertical angles, they are congruent. We can now see that two sides and the included angle of $\triangle ABC$ are congruent to two sides and the included angle of $\triangle ADE$. See Figure G.25. By SAS, $\triangle ABC$ and $\triangle ADE$ are congruent. Since side $\overline{ED}$ corresponds to side $\overline{BC}$ and corresponding parts of congruent triangles are congruent, we have $\overline{ED} \cong \overline{BC}$.

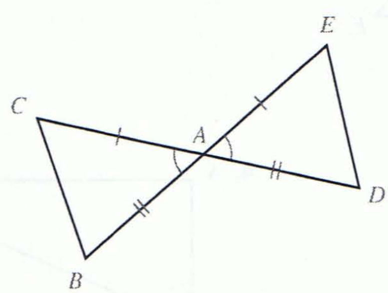



**Figure G.25**

## EXAMPLE 5

In Figure G.26, $\triangle ABC$ is isosceles with $\overline{AC} \cong \overline{BC}$ and $\overline{AB} \perp \overline{CD}$. Explain why $\angle ACD \cong \angle BCD$ and $\overline{AD} \cong \overline{DB}$.

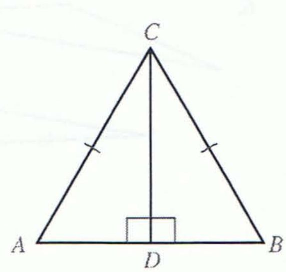



**Figure G.26**

**Solution** We are given that $\overline{AC} \cong \overline{BC}$. It follows that $\angle A \cong \angle B$, since they are the base angles of an isosceles triangle. Since $\overline{AB} \perp \overline{CD}$, $\angle ADC$ and $\angle BDC$ are both right angles, and therefore $\angle ADC \cong \angle BDC$. See Figure G.27(**a**). Let's now focus on the two triangles $\triangle ADC$ and $\triangle BDC$. Since two angles of $\triangle ADC$ are congruent to two angles of $\triangle BDC$, by our discussion in the last section the third angles of each triangle must be congruent to each other. Hence, we have $\angle ACD \cong \angle BCD$.

Since we have two angles and the included side of $\triangle ADC$ congruent to two angles and the included side of $\triangle BDC$, by ASA we have $\triangle ADC \cong \triangle BDC$. See Figure G.27(**b**). Therefore $\overline{AD} \cong \overline{DB}$, since they are corresponding parts of congruent triangles.

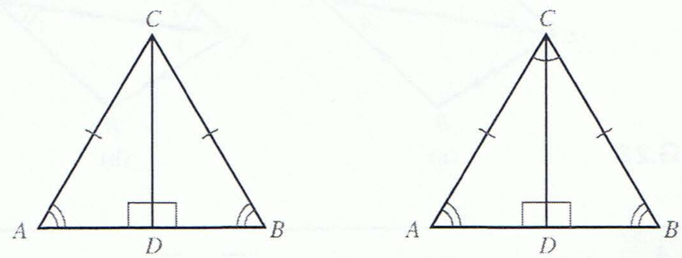
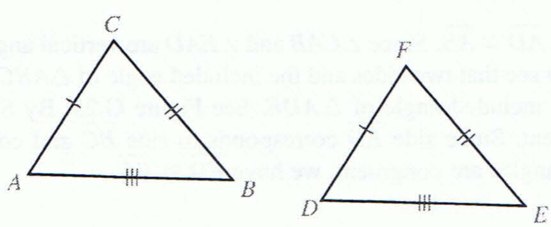



**Figure G.27** (a) (b)

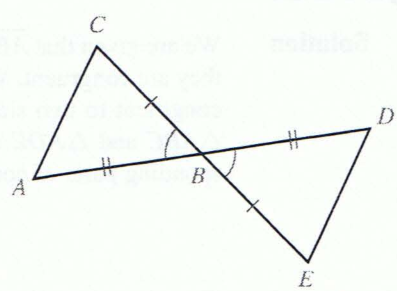
## EXERCISES G.3

*In Exercises* 1–6, *explain why the two triangles are congruent.*

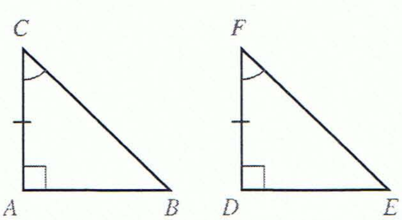
**1.**



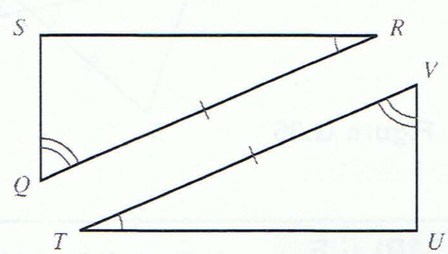

**2.**



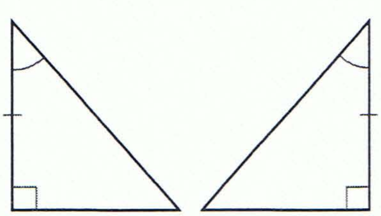

**3.**



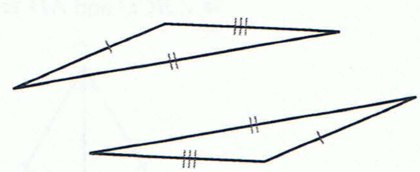

**4.**




**5.**

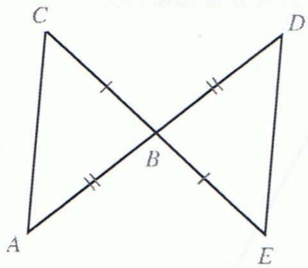


**6.**

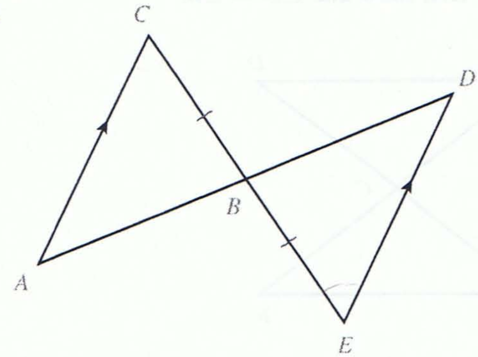


*In Exercises 7–14, explain why the two triangles are congruent.*

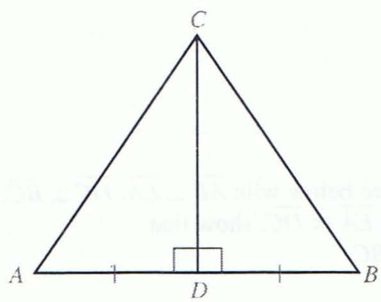
**7.**




**8.**

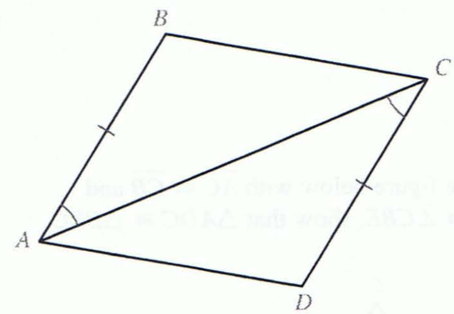



**9.**

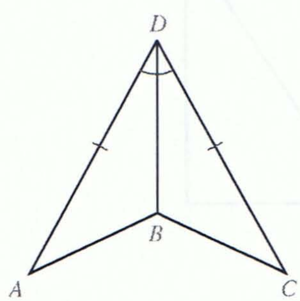



$\Delta ACD \cong \Delta BCD$

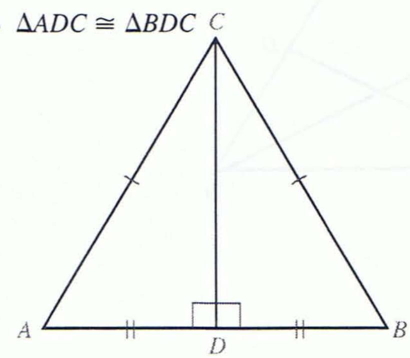
**10.**



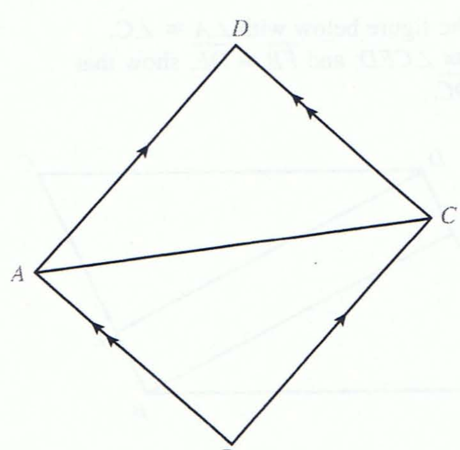

**11.**



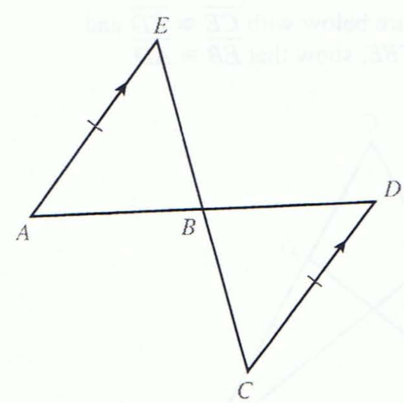

**12.** $\Delta ADC \cong \Delta BDC$



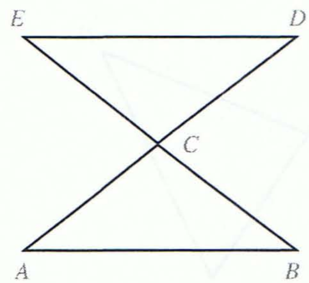

**13.**



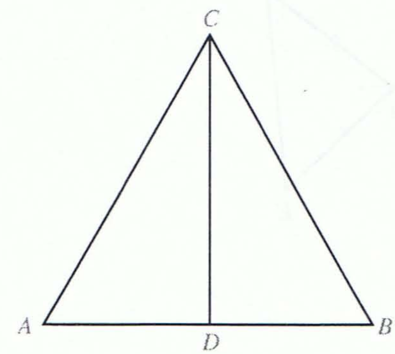

**14.**



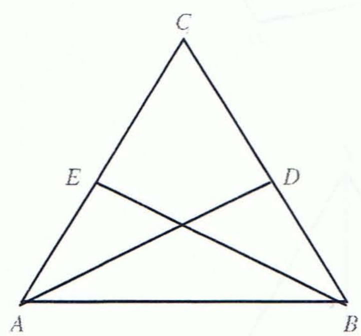
**15.** Given the figure below with $\overline{CE} \cong \overline{CA}$ and $\angle E \cong \angle A$, show that $\angle D \cong \angle B$.



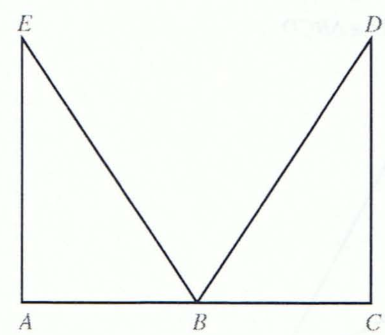

**16.** Given the figure below with $\angle ACD \cong \angle BCD$ and $\overline{CD} \perp \overline{AB}$, show that $\triangle ACB$ is isosceles.



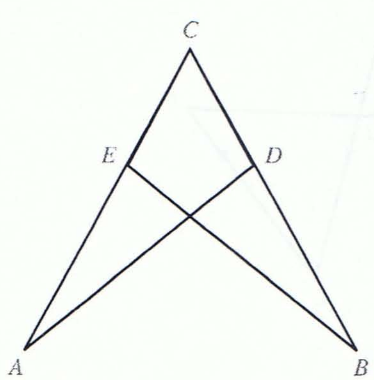

**17.** Given the figure below with $\overline{AC} \cong \overline{CB}$ and $\angle CAD \cong \angle CBE$, show that $\triangle ADC \cong \triangle BEC$.



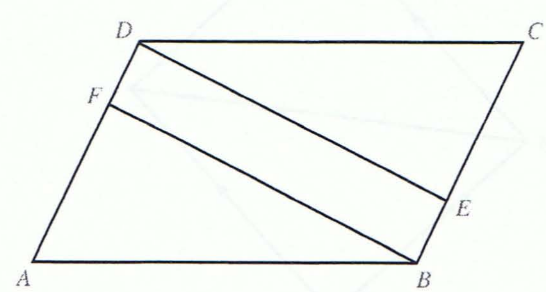

**18.** Given the figure below with $\overline{AB} \perp \overline{EA}$, $\overline{DC} \perp \overrightarrow{BC}$, $\overline{AB} \cong \overline{BC}$, and $\overline{EA} \cong \overline{DC}$, show that $\angle EBA \cong \angle DBC$.




**19.** Given the figure below with $\overline{CE} \cong \overline{CD}$ and $\angle CAD \cong \angle CBE$, show that $\overline{EB} \cong \overline{AD}$.

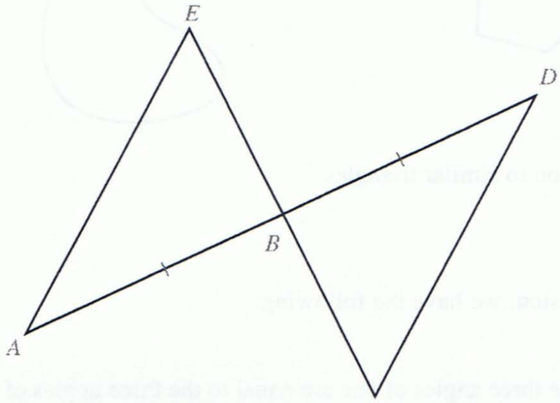



**20.** Given the figure below with $\angle A \cong \angle C$, $\angle AFB \cong \angle CED$, and $\overline{FB} \cong \overline{DE}$, show that $\overline{AB} \cong \overline{DC}$.



*In Exercise 21–24, determine what additional information is necessary to show that the triangles are congruent by the given theorems.*

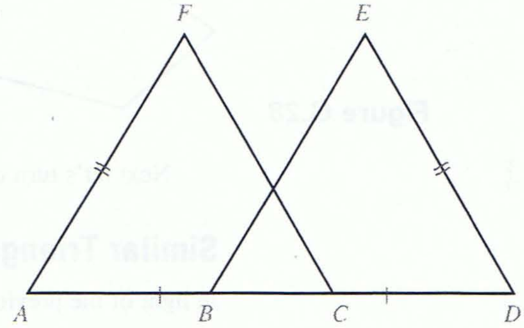
**21.**




$\triangle ABE \cong \triangle CBD$

**(a)** By SAS
**(b)** By ASA

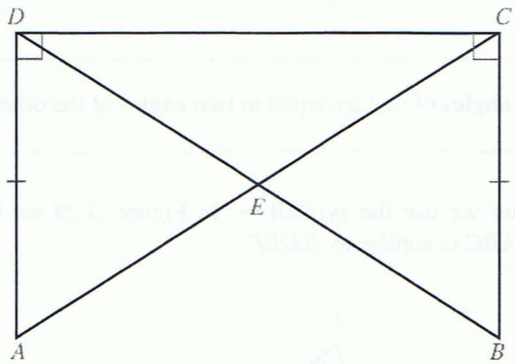
**22.**




$\triangle ACF \cong \triangle BDE$

**(a)** By SSS
**(b)** By SAS

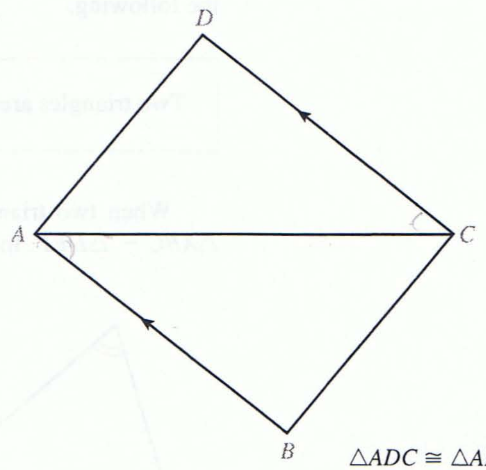
**23.**




$\triangle ADC \cong \triangle BCD$

**(a)** By ASA
**(b)** By SAS

**24.**




$\triangle ADC \cong \triangle ABC$

**(a)** By ASA
**(b)** By SAS

## G.4 Similarity

We are familiar with instruments that enlarge or reduce the observed size of objects. A magnifying glass, a telescope, and a movie projector are familiar examples of instruments that enlarge the observed size of an object. Scale drawings, maps, and architectural plans are examples of images that are obtained by shrinking the observed size of an object. What all these instruments and situations have in common is that the enlarged or shrunken image maintains the same shape as the original object, but not necessarily the same size.

When two geometric figures have the same shape we say that they are *similar.* For example, Figure G.28 on page 556 illustrates two pairs of similar geometric figures. In particular, when the geometric figures are polygons (figures formed by straight line segments), having the same shape means having the same angles.

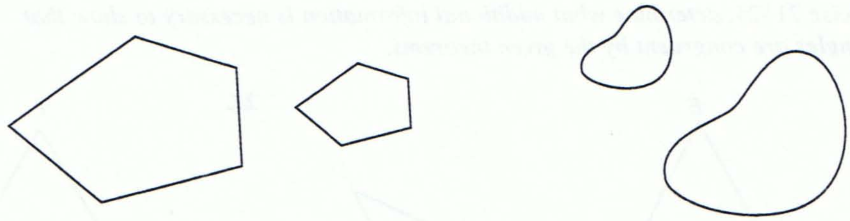


Figure G.28

Next let's turn our attention to similar triangles.

## Similar Triangles

In light of the previous discussion, we have the following.

**DEFINITION** Two triangles are ***similar*** if the three angles of one are equal to the three angles of the other.

As we noted in Section G.2, if two angles of one triangle are equal to two angles of another triangle, then the third angle of each must be equal as well. Thus, we have the following.

> Two triangles are similar if two angles of one are equal to two angles of the other.

When two triangles are similar we use the symbol $\sim$. In Figure G.29 we have $\triangle ABC \sim \triangle DEF$, meaning that $\triangle ABC$ is similar to $\triangle DEF$.

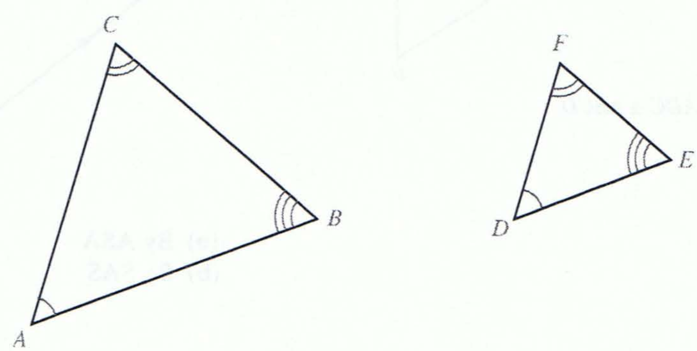



Figure G.29

As with congruent triangles, the pairs of congruent angles are called ***corresponding angles.*** In Figure G.30, $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$. With similar triangles we refer to the ***corresponding sides*** as the sides opposite the congruent angles. In Figure G.30, $\triangle ABC \sim \triangle A'B'C'$: The side of length 3 corresponds to the side of length 6 (since they are both opposite equal angles). Similarly, the other pairs of corresponding sides are 5 and 10, and 7 and 14.

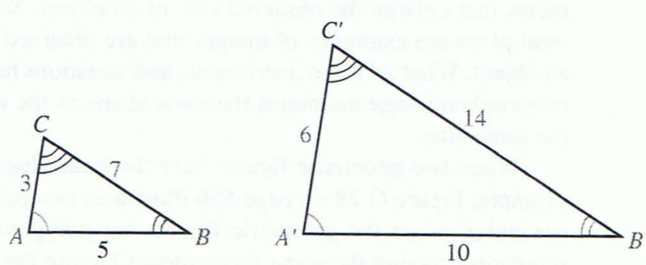



Figure G.30

Note that the corresponding sides are in proportion:

$$\frac{3}{6} = \frac{5}{10} = \frac{7}{14}$$

All similar triangles share this property, which is stated formally in the following theorem.

**THEOREM G.5** The corresponding sides of similar triangles are in proportion.

## EXAMPLE 1

In Figure G.31, find $x$.

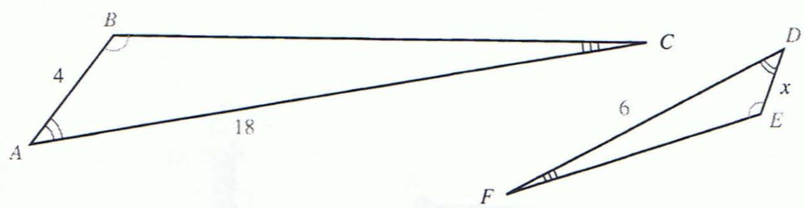



**Figure G.31**

**Solution** Since the three angles of $\triangle ABC$ are congruent to the three angles of $\triangle DEF$, $\triangle ABC \sim \triangle DEF$. The side we need to find is $|\overline{DE}| = x$; its corresponding side in $\triangle ABC$ is $\overline{AB}$. (Note that they are the sides opposite equal angles.)

We set up the proportion as follows:

$$\frac{|\overline{DE}|}{|\overline{AB}|} = \frac{|\overline{DF}|}{|\overline{AC}|}$$ *Corresponding side from △DEF / Corresponding side from △ABC*

*Among other correct proportions we could write is $\frac{x}{6} = \frac{4}{18}$.*

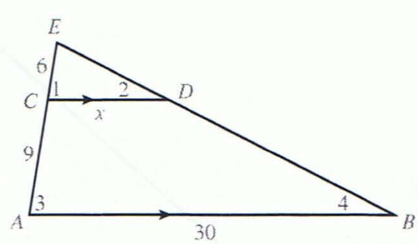
$$\frac{x}{4} = \frac{6}{18}$$ *Solve for x; multiply each side by 4 to get*

$$4 \cdot \frac{x}{4} = 4 \cdot \frac{6}{18}$$

$$x = \frac{24}{18} = \frac{4}{3} = \boxed{1\frac{1}{3}}$$

## EXAMPLE 2

Refer to the figure below. Given that $\overline{AB}$ is parallel to $\overline{CD}$, find the value of $x$.




**Solution** Since $\overline{AB}$ is parallel to $\overline{CD}$, $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$ (they are corresponding angles of the parallel lines). Therefore, $\triangle ABE$ is similar to $\triangle CDE$ (why?), and the corresponding sides are in proportion. We then have

$$\frac{x}{|\overline{AB}|} = \frac{|\overline{CE}|}{|\overline{AE}|}$$ *Note that on both sides of the proportion the ratio is a side one triangle to the corresponding side of the other triangle.*

$$\frac{x}{30} = \frac{6}{15}$$ *Multiply both sides of the equation by 30.*

$$x = \frac{30 \cdot 6}{15}$$

$$\boxed{x = 12}$$

## EXAMPLE 3

Matt measures the height of a tree in the following way. He stands 50 feet away from the tree and asks his friend Al to walk toward him, starting from the tree. Matt lies down on the (level) ground and watches Al walk toward him. As soon as the top of Al's head is in the same line of sight as the top of the tree, Matt tells Al to stop. See Figure G.32. Matt then measures the distance from where he was lying to where Al stopped. If this distance is 6 feet and he knows that Al is 5′6″, how tall is the tree (to the nearest foot)?

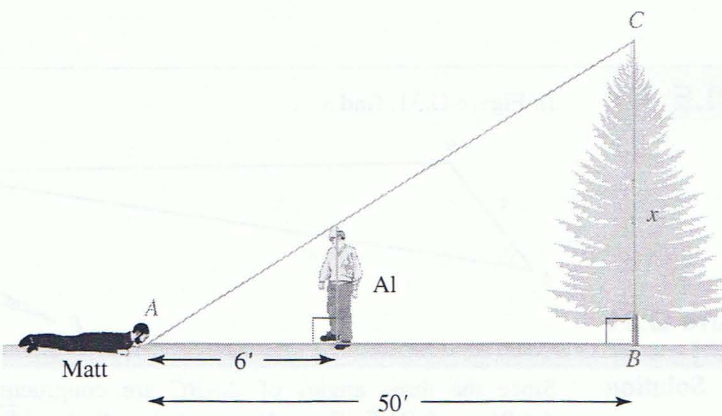



**Figure G.32**

**Solution** You can see by the figure that there is a right triangle with the tree as one side, and a right triangle with Al as one side. Both right triangles share angle *A*, the angle formed by the ground and Matt's line of sight. Each triangle also has a right angle. This means that two of the angles of one triangle are equal to two of the angles of the other, and the two right triangles are therefore similar. Since the triangles are similar, the sides are in proportion. We are trying to find the height of the tree, so we form a proportion using the height of the tree and the sides we are given as follows (see Figure G.33):

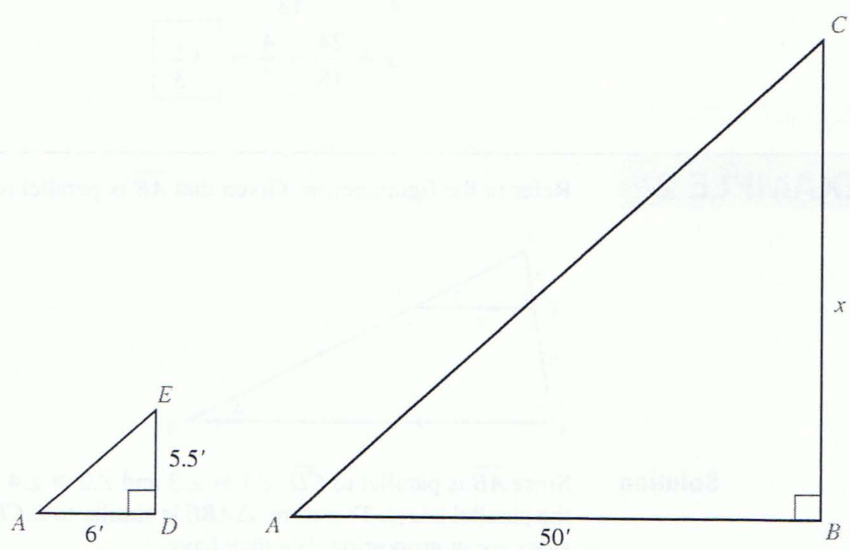



**Figure G.33**

$$\frac{\textit{Height of tree}}{\textit{Al's height}} = \frac{\textit{Distance between Matt and the tree}}{\textit{Distance between Matt and Al}}$$

$$\frac{x}{5.5} = \frac{50}{6}$$

*We changed Al's height 5′6″ into 5.5 ft. The height of the tree is x. Solve for x.*

$$x = \frac{50}{6}(5.5) \approx \boxed{\text{46 ft (to the nearest foot)}}$$

## EXAMPLE 4

In Figure G.34, find $x$.

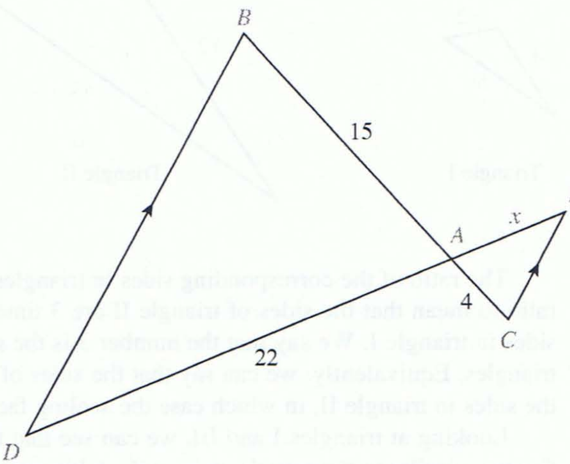



**Figure G.34**

**Solution** From Figure G.34 we see that $BD \parallel EC$. Line segment $\overline{BC}$ is a transversal cutting across the parallel lines. Since $\angle B$ and $\angle C$ are alternate interior angles, they are congruent: $\angle B \cong \angle C$, see Figure G.35. $\angle 1$ is congruent to $\angle 2$ since they are vertical angles. This is enough to establish that $\triangle ABD$ and $\triangle ACE$ are similar. Again, see Figure G.35.

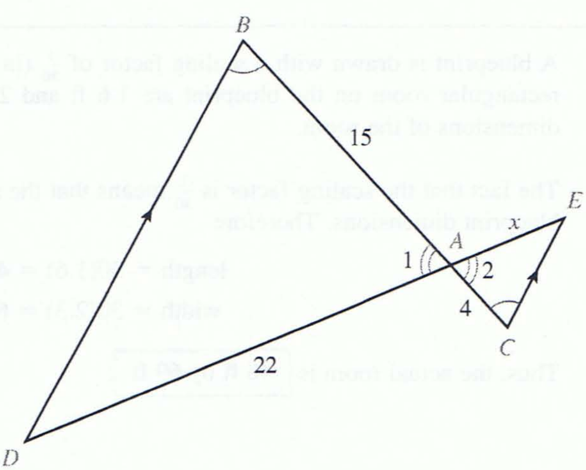



**Figure G.35**

Since $\triangle ABD \sim \triangle ACE$, their corresponding sides are proportional.

$$\frac{|\overline{AE}|}{|\overline{AD}|} = \frac{|\overline{AC}|}{|\overline{AB}|} \qquad \frac{\textit{Corresponding side from } \triangle ACE}{\textit{Corresponding side from } \triangle ABD}$$

$$\frac{x}{22} = \frac{4}{15} \qquad \textit{Solve for } x\textit{; multiply each side by 22 to get}$$

$$x = \frac{4}{15}(22)$$

$$x = \frac{88}{15} = \boxed{5\frac{13}{15}}$$

## Scaling Factors

Let's look at the idea of similarity from a slightly different perspective. Consider the three similar triangles in Figure G.36.

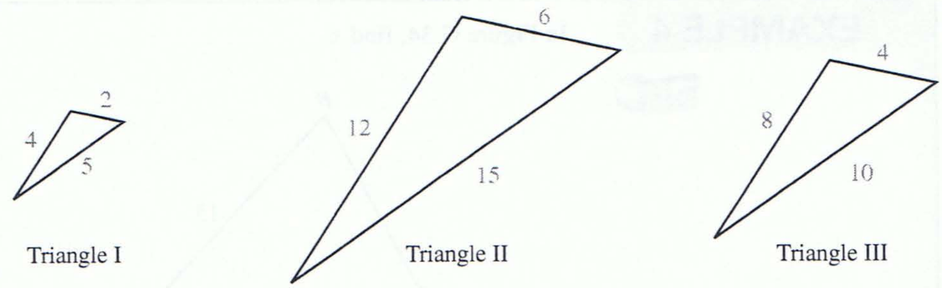



**Figure G.36**

The ratio of the corresponding sides in triangles I and II is $\frac{1}{3}$. We can interpret this ratio to mean that the sides of triangle II are 3 times the lengths of the corresponding sides in triangle I. We say that the number 3 is the ***scaling factor*** for these two similar triangles. Equivalently, we can say that the sides of triangle I are $\frac{1}{3}$ times the length of the sides in triangle II, in which case the scaling factor is $\frac{1}{3}$.

Looking at triangles I and III, we can see that the scaling factor is 2. This scaling factor is similar to the way the power of a telescope or a pair of binoculars is described. If we are told that a telescope has a "100 power" lens, this means that the observed size of the object through the telescope is 100 times larger than the size of the object as observed with the naked eye. In other words, the image size has been scaled up by a factor of 100. Similarly, if a scale model of an airplane is built on a scale of $\frac{1}{60}$, this means that 1 inch on the model represents 60 inches (or 5 feet) of the actual airplane.

**EXAMPLE 5**

A blueprint is drawn with a scaling factor of $\frac{1}{30}$ (in feet). If the length and width of a rectangular room on the blueprint are 1.6 ft and 2.3 ft, respectively, find the actual dimensions of the room.

**Solution**

The fact that the scaling factor is $\frac{1}{30}$ means that the actual dimensions are 30 times the blueprint dimensions. Therefore

$$\text{length} = 30(1.6) = 48 \text{ ft}$$
$$\text{width} = 30(2.3) = 69 \text{ ft}$$

*Alternatively, we could have found the length using the proportion* $\frac{1.6}{length} = \frac{1}{30}$

Thus, the actual room is $\boxed{48 \text{ ft by } 69 \text{ ft}}$.

## Two Special Triangles

Two special triangles play important roles in mathematics. They are the ***isosceles right triangle*** and the **30°–60° *right triangle.***

***The Isosceles Right Triangle*** Figure G.37 shows an isosceles right triangle. Note that because the legs are equal, the base angles must be equal, and since the base angles must have a sum of 90°, they must be 45° each. We have labeled each leg $s$ and the hypotenuse $x$. We solve for $x$ by using the Pythagorean theorem.

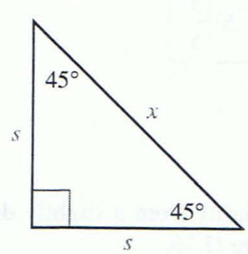



**Figure G.37**
An isosceles right triangle

$$x^2 = s^2 + s^2$$

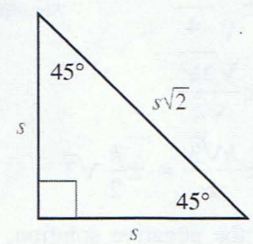
$$x^2 = 2s^2 \qquad \textit{Take square roots to get}$$

$$x = \pm\sqrt{2s^2} \qquad \textit{Simplify the radical. Since } s \textit{ is positive, } \sqrt{s^2} = s.$$

$$x = \pm\sqrt{s^2}\,\sqrt{2} = \pm s\sqrt{2}$$

Since $x$ is a length, we reject the negative solution, so $x = s\sqrt{2}$. We have just derived the following:

**The Isosceles (45°) Right Triangle**




In words, the diagram says that in a 45° right triangle, the legs are equal and the hypotenuse is $\sqrt{2}$ times the length of the leg.

***The 30°–60° Right Triangle*** Figure G.38(**a**) shows a 30°–60° right triangle. We label the hypotenuse $h$. If we duplicate the triangle as indicated by the dotted lines in Figure G.38(**b**), we can see that $\triangle ABD$ is equilateral (because each angle is 60°), so that $|\overline{AD}|$ is also $h$ and $|\overline{AC}|$ must be $\frac{h}{2}$.

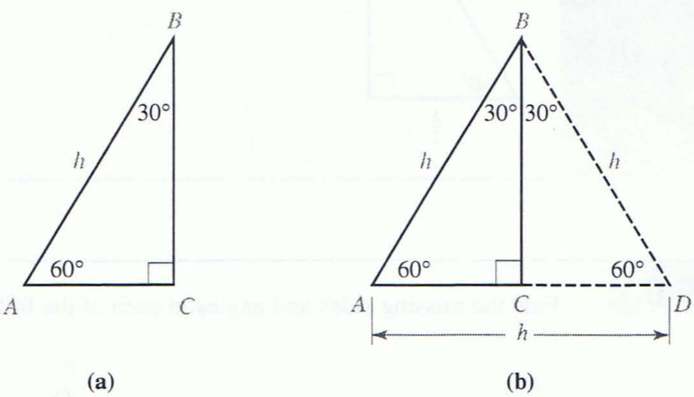



**Figure G.38**

In Figure G.39 we redraw Figure G.38(**a**). We have labeled the hypotenuse $h$, $|\overline{AC}|$ as $\dfrac{h}{2}$, and the unknown side $|\overline{BC}|$ as $x$. We find $x$ by again using the Pythagorean theorem.

*Why is the name "30°–60° right triangle" actually redundant?*

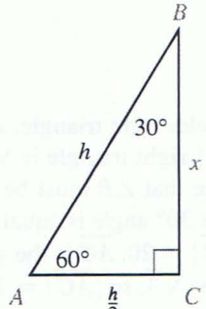



**Figure G.39**

$$x^2 + \left(\frac{h}{2}\right)^2 = h^2$$

$$x^2 + \frac{h^2}{4} = h^2 \qquad \textit{Isolate } x^2. \textit{ Subtract } \frac{h^2}{4} \textit{ from each side.}$$

$$x^2 = h^2 - \frac{h^2}{4} \qquad \textit{Combine.}$$

$$x^2 = \frac{3h^2}{4} \qquad \textit{Take square roots.}$$

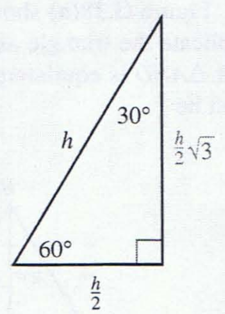
$$x = \pm\sqrt{\frac{3h^2}{4}} \qquad \textit{Simplify the radical.}$$

$$x = \pm\frac{\sqrt{3h^2}}{\sqrt{4}}$$

$$x = \pm\frac{h\sqrt{3}}{2} = \pm\frac{h}{2}\sqrt{3}$$

As before, we reject the negative solution, so $x = \frac{h}{2}\sqrt{3}$. We have derived the following:

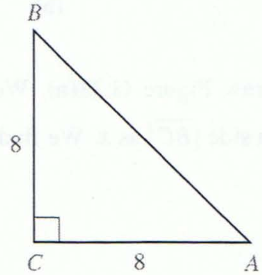
## The 30°–60° Right Triangle



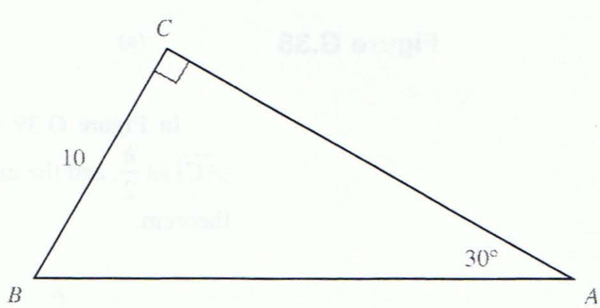

In words, the diagram says that in a 30°–60° right triangle the side opposite the 30° angle is one-half the hypotenuse, and the side opposite the 60° angle is one-half the hypotenuse times $\sqrt{3}$.

## EXAMPLE 6

Find the missing sides and angles in each of the following triangles.




**(a)**




**(b)**

*In words, describe the relationships among the sides of an isosceles right triangle.*

### Solution

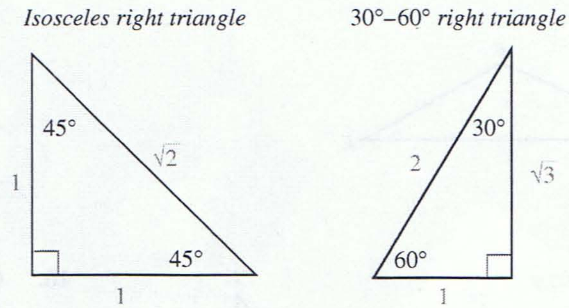
**(a)** Since the triangle is an isosceles right triangle, $\angle A = \angle B = 45°$. Since the hypotenuse of a 45° right triangle is $\sqrt{2}$ times the leg, $|\overline{AB}| = 8\sqrt{2}$.

**(b)** From the diagram we can see that $\angle B$ must be 60°. Since this is a 30°–60° right triangle, the side opposite the 30° angle is equal to half the hypotenuse. Therefore, $|\overline{BC}| = 10 = \frac{1}{2}|\overline{AB}|$, so $|\overline{AB}| = 20$. $\overline{AC}$ is the side opposite 60° and is, therefore, one-half the hypotenuse times $\sqrt{3}$, so $|\overline{AC}| = 10\sqrt{3}$.

*In words, describe the relationships among the sides of a 30°–60° right triangle.*

The simplest examples of these two special triangles are obtained by choosing the leg of the 45° right triangle to be 1 and by choosing the hypotenuse of the 30°—60° right triangle to be 2. We then obtain the prototypes illustrated in the following box.

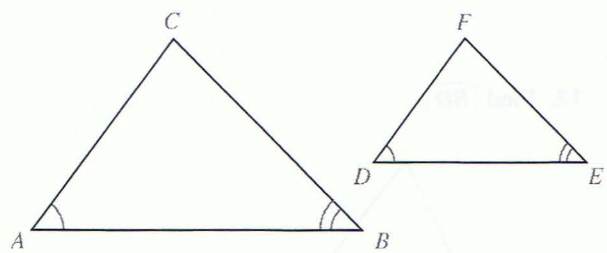
**Prototypes for the Isosceles and 30°–60° Right Triangles**




## EXERCISES G.4

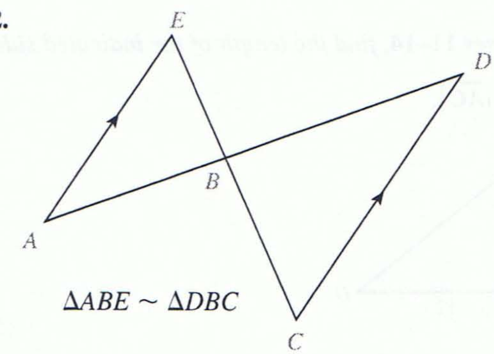
*In Exercises 1–6, explain why the given pairs of triangles are similar. Identify the corresponding angles.*

**1.**



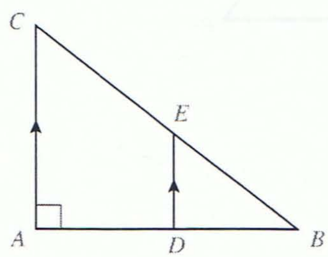

$\Delta ACB \sim \Delta DFE$

**2.**



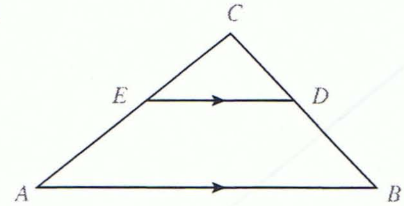

$\Delta ABE \sim \Delta DBC$

**3.**



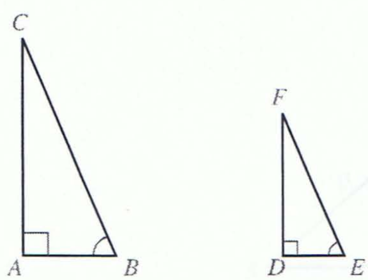

$\Delta CAB \sim \Delta EDB$

**4.**



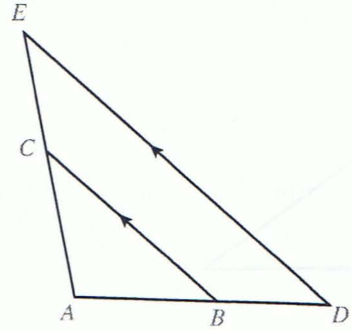

$\Delta ABC \sim \Delta EDC$

**5.**




$\Delta ABC \sim \Delta DEF$

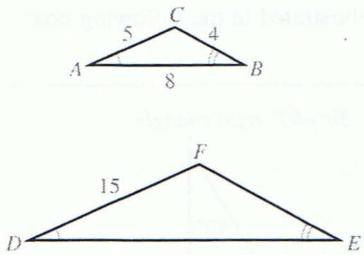
**6.**



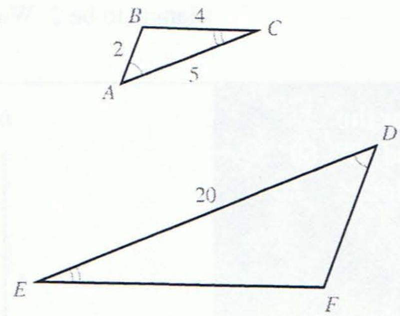

$\Delta ABC \sim \Delta ADE$

*In Exercises 7–10, $\triangle ABC$ is similar to $\triangle DEF$. Find the missing sides in $\triangle DEF$.*

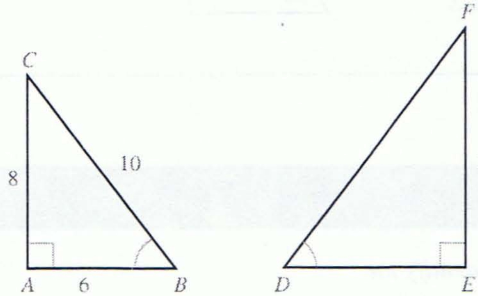
**7.**



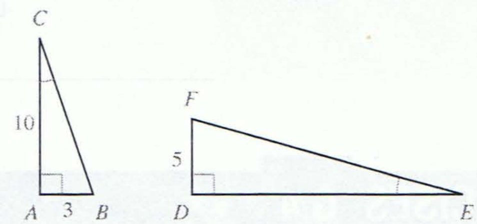

**8.**




**9.**

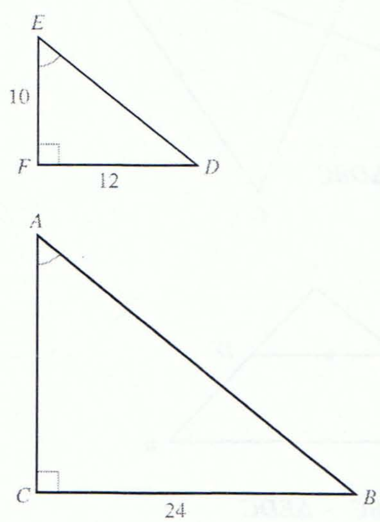



**10.**

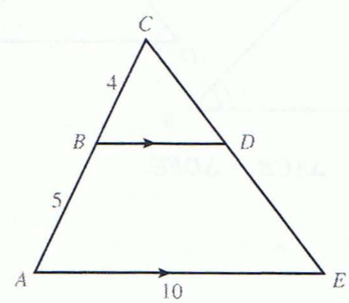



*In Exercises 11–14, find the length of the indicated side.*

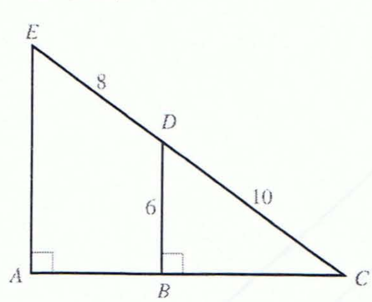
**11.** Find $|\overline{AC}|$.



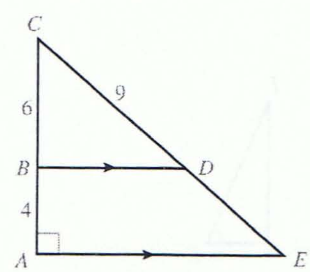

**12.** Find $|\overline{BD}|$.




**13.** Find $|\overline{EA}|$.

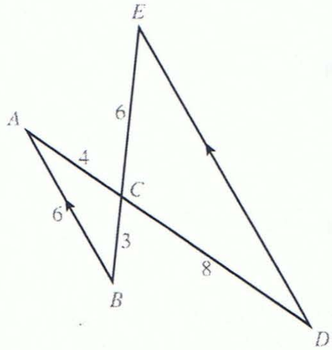



**14.** Find $|\overline{DE}|$.

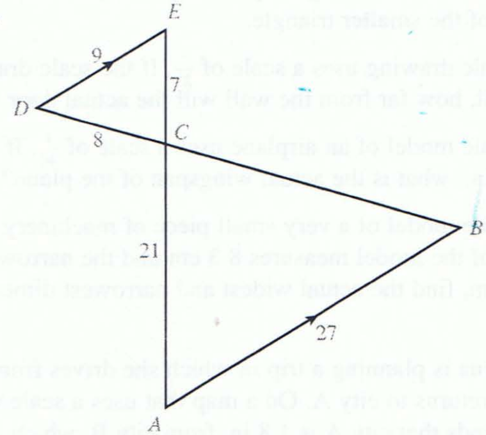


*In Exercises* 15–18, *find the missing sides.*

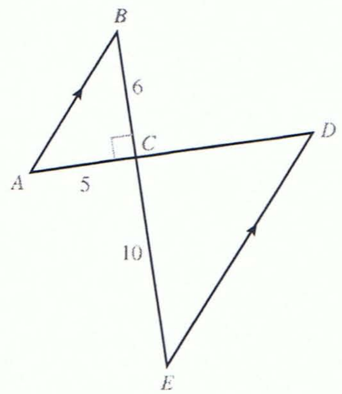
**15.**



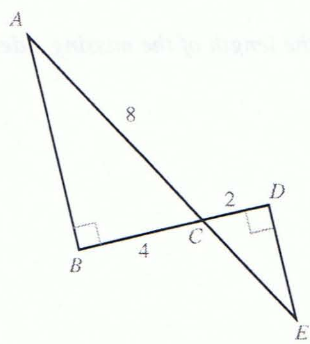

**16.**



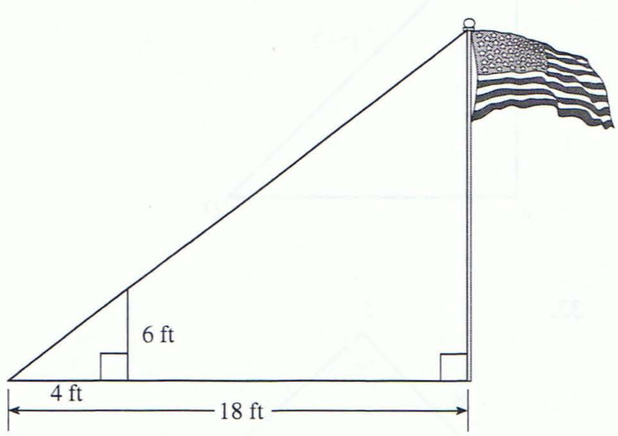

**17.**




**18.**




**19.** Suppose that a man 6 ft tall casts a shadow 4 ft long. Determine the height of a flagpole that casts a shadow 18 ft long. See the accompanying figure.




**20.** Suppose that a bush 3 ft tall casts a shadow 5 ft long. Determine the length of the shadow cast by a tree 20 ft tall.

*In Exercises* 21–26, *round your answer to the nearest tenth where necessary.*

**21.** The corresponding sides of two similar triangles are in the ratio of 4 to 7. If a side of the smaller triangle is 5.8 cm, find the length of the corresponding side of the larger triangle.

**22.** The corresponding sides of two similar geometric figures are in the ratio of 9 to 4. If a side of the larger figure is 15.3 m, find the length of the corresponding side of the smaller triangle.

**23.** A scale drawing uses a scale of $\frac{1}{25}$. If the scale drawing places a door 1.4 in. from a wall, how far from the wall will the actual door be placed?

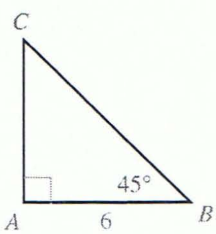
**24.** A scale model of an airplane uses a scale of $\frac{1}{40}$. If the model exhibits a wingspan of 8 in., what is the actual wingspan of the plane?

**25.** A scale model of a very small piece of machinery uses a scale of $\frac{28}{1}$. If the widest part of the model measures 8.3 cm and the narrowest part of the model measures 3.2 cm, find the actual widest and narrowest dimensions of the actual machine part.

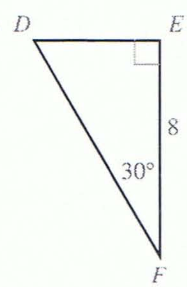
**26.** Shawna is planning a trip in which she drives from city A to city B to city C and then returns to city A. On a map that uses a scale where 1 in. represents 50 miles, she finds that city A is 1.8 in. from city B, which is 2.2 in. from city C, which is 0.9 in. from city A. Find the actual driving distance for this trip.

*In Exercises 27–34, find the length of the missing sides of the given right triangles.*

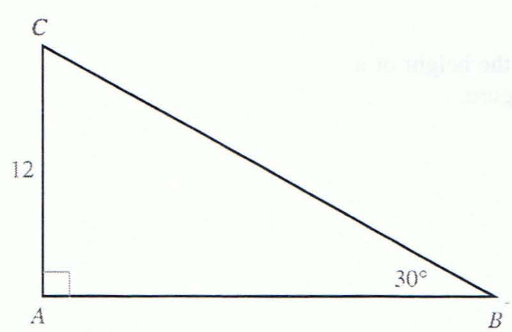
**27.**



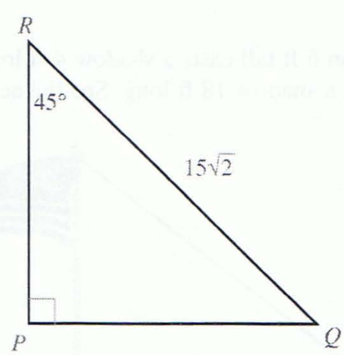

**28.**



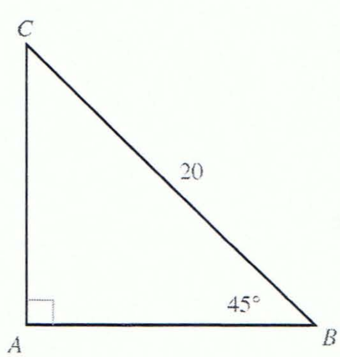

**29.**



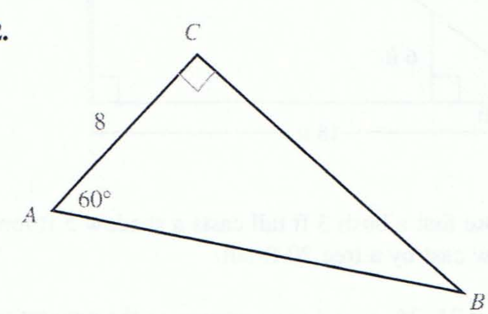

**30.**



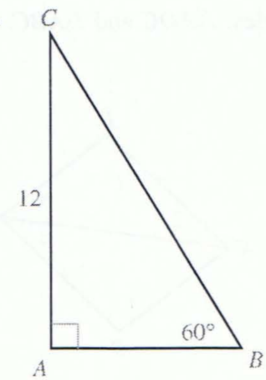

**31.**



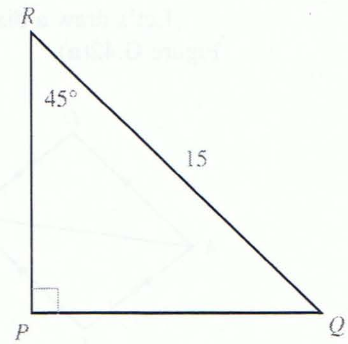

**32.**



**33.**




**34.**




# G.5 Quadrilaterals

We have already seen that the sum of the angles of a triangle is 180°. As we can see in Figure G.40, any quadrilateral can be divided into two triangles. In $\triangle ABD$, $\angle 1 + \angle 2 + \angle 3 = 180°$, and in $\triangle BCD$, $\angle 4 + \angle 5 + \angle 6 = 180°$. But all these angles together are the angles of the quadrilateral; therefore, we can see that

$$\angle A + \angle B + \angle C + \angle D = 360°$$

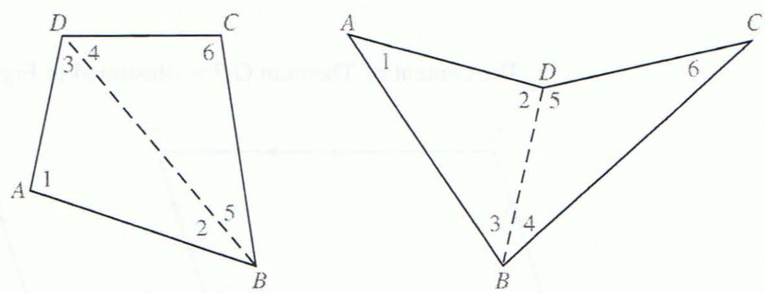



**Figure G.40**

We have just proved the following theorem.

**THEOREM G.6** The sum of the angles of a quadrilateral is 360°.

## Parallelograms

A quadrilateral in which both pairs of opposite sides are parallel is called a ***parallelogram.***

A ***diagonal*** is a line segment that joins two nonadjacent vertices.

Let's examine the parallelogram *ABCD* in Figure G. 41.

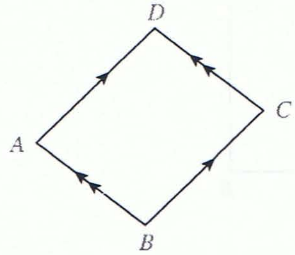



**Figure G.41**

Let's draw a diagonal from $A$ to $C$ to form two triangles: $\triangle ADC$ and $\triangle ABC$. See Figure G.42(**a**).

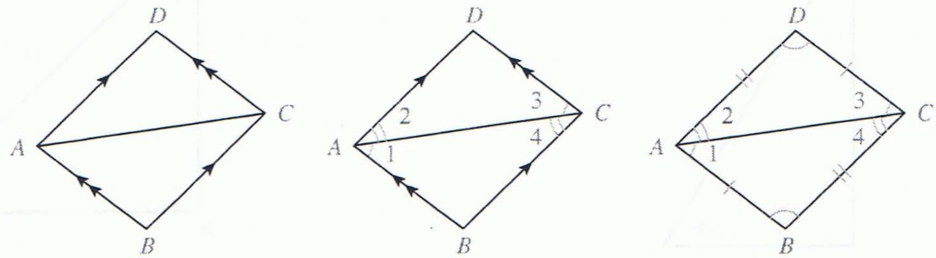



**Figure G.42** (a) (b) (c)

Viewing $\overline{AC}$ as a transversal cutting across the parallel line segments $\overline{AB}$ and $\overline{CD}$, we see that $\angle 1 \cong \angle 3$, since they are alternate interior angles to this pair of parallel lines. See Figure G.42. On the other hand, viewing $\overline{AC}$ as a transversal cutting across the parallel line segments $\overline{AD}$ and $\overline{BC}$, we see that $\angle 2 \cong \angle 4$, since they are alternate interior angles to this pair of parallel lines (Figure G.42(**b**)). Both triangles share the common side $\overline{AC}$ (Figure G.42.(**c**)), and by ASA, the two triangles are congruent. Since corresponding parts of congruent triangles are congruent, $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$; and $\angle B \cong \angle D$. Since $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$, we have $\angle 1 + \angle 2 = \angle 3 + \angle 4$. This means that $\angle DAB \cong \angle BCD$. We have the following:

**THEOREM G.7**

1. The opposite angles of a parallelogram are equal.
2. The opposite sides of a parallelogram are equal.

The content of Theorem G.7 is illustrated in Figure G.43.

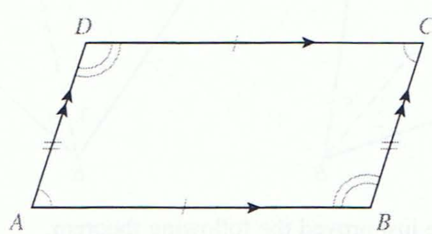



**Figure G.43**
The opposite sides and angles of parallelogram $ABCD$ are equal.

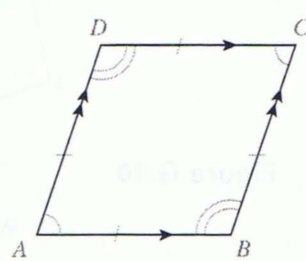



**Figure G.44**
Rhombus $ABCD$

If all four sides of a parallelogram are equal, it is called a ***rhombus*** (see Figure G.44).

If a parallelogram contains a right angle (and therefore it follows that all its angles must be right angles), it is called a ***rectangle*** (see Figure G.45(**a**)).

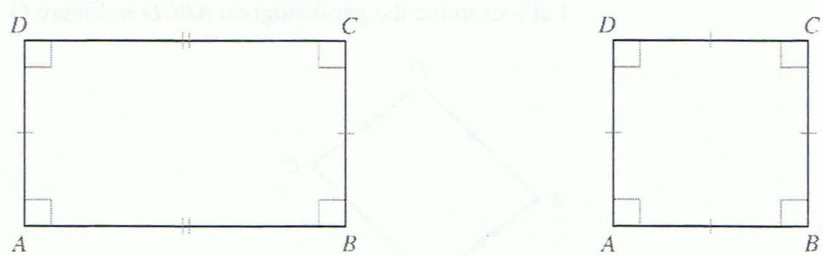



**Figure G.45** (**a**) Rectangle $ABCD$ (**b**) Square $ABCD$

If the adjacent sides of a rectangle are equal (and therefore all four sides are equal), it is called a ***square*** (see Figure G.45(**b**)).

## EXAMPLE 1

Show that the diagonals of a parallelogram bisect each other.

**Solution** We start by drawing a picture of a parallelogram $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at $E$. See Figure G.46.

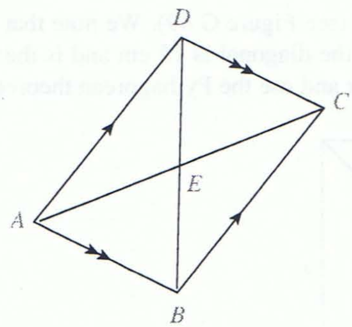



**Figure G.46**

This time we focus on $\triangle AEB$ and $\triangle CED$. Viewing $\overline{AC}$ as a transversal cutting across the parallel line segments $\overline{AB}$ and $\overline{DC}$, we see that $\angle 1 \cong \angle 2$ since they are alternate interior angles to this pair of parallel lines (Figure G.47(**a**)). Viewing $\overline{DB}$ as a transversal cutting across the same parallel line segments, we see that $\angle 3 \cong \angle 4$ since they are alternate interior angles to this pair of parallel lines (Figure G.47(**b**)).

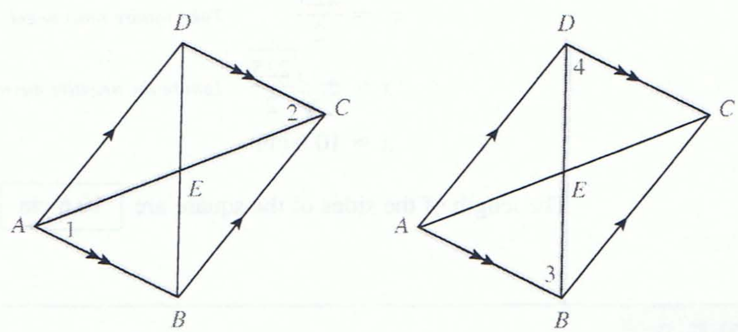



**Figure G.47** (a) (b)

Since opposite sides of a parallelogram are congruent (Theorem G.7), by ASA, we have $\triangle AEB \cong \triangle CED$. See Figure G.48(**a**). $\overline{DE} \cong \overline{BE}$, since they are corresponding parts of congruent triangles. This shows that diagonal $\overline{AC}$ divides diagonal $\overline{DB}$ into equal parts. Hence, diagonal $\overline{AC}$ bisects diagonal $\overline{DB}$. See Figure G.48(**b**).

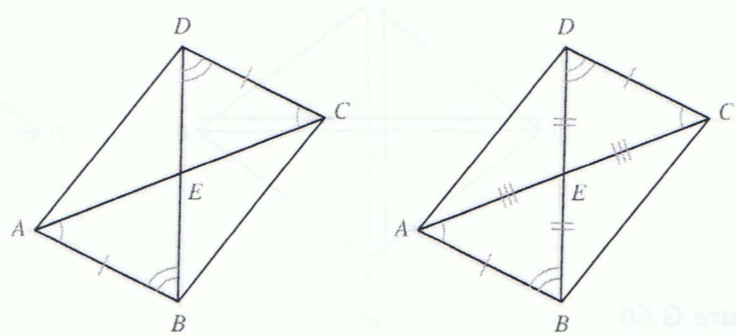



**Figure G.48** (a) (b)

$\overline{AE} \cong \overline{EC}$, since they are also corresponding parts of congruent triangles. This shows that diagonal $\overline{DB}$ divides diagonal $\overline{AC}$ into equal parts. Hence, diagonal $\overline{DB}$ bisects diagonal $\overline{AC}$. See Figure G.48(**b**).

## EXAMPLE 2

To the nearest tenth of a centimeter, find the length of the side of a square with diagonal 15 cm.

**Solution** We draw the figure (see Figure G.49). We note that a diagonal divides the square into two right triangles. the diagonal is 15 cm and is the hypotenuse of each right triangle. We label each side $x$ and use the Pythagorean theorem to find $x$, the length of the sides:

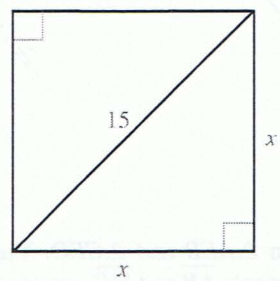



**Figure G.49**

$$x^2 + x^2 = 15^2$$
$$2x^2 = 225$$
$$x^2 = \frac{225}{2} \quad \textit{Take square roots to get}$$
$$x = \pm\sqrt{\frac{225}{2}} \quad \textit{Ignore the negative answer. We use a calculator to get}$$
$$x \approx 10.6 \text{ cm}$$

The length of the sides of the square are $\boxed{10.6 \text{ cm}}$, rounded to the nearest tenth.

## EXAMPLE 3

In a rhombus, the diagonals are perpendicular bisectors of each other. Find the length of the sides of a rhombus with diagonals 12 in. and 16 in.

**Solution** Again we start by drawing a figure (Figure G.50(**a**)). We draw the diagonals of the rhombus so that they bisect each other and meet at right angles.

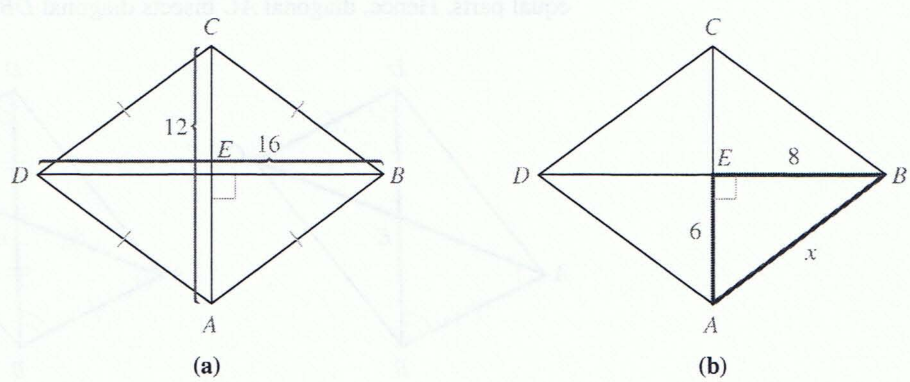



**Figure G.50**

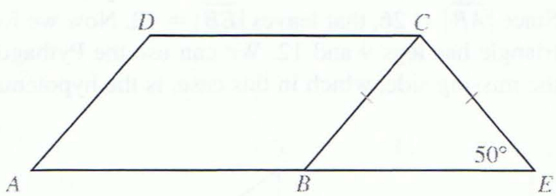
We note that now we have four right triangles. Let's examine $\triangle ABE$. Since the diagonals of the rhombus bisect each other, each leg of $\triangle ABE$ is half the length of each diagonal; therefore the legs of $\triangle ABE$ are 6 in. and 8 in. See Figure G.50(**b**). (We note that all four triangles are congruent to each other since they each have the same-size legs, and all sides of a rhombus are equal.) The side of the rhombus is the hypotenuse of $\triangle ABE$. We label the hypotenuse $x$ and use the Pythagorean theorem to find it:

$$6^2 + 8^2 = x^2$$
$$36 + 64 = x^2$$
$$100 = x^2 \quad \textit{Take square roots to get}$$
$$10 = x \quad \textit{Use the positive root.}$$

The length of the sides of the rhombus is 10 inches.

## EXAMPLE 4

Given the following diagram, find the measures of the four angles in parallelogram $ABCD$.




**Solution**

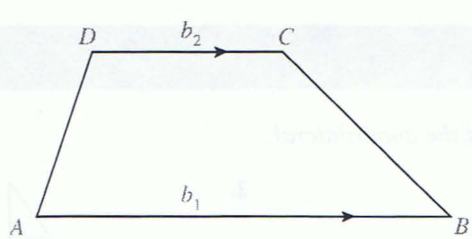
Since $\triangle BEC$ is isosceles, $\angle CBE = 50°$ (because the base angles of an isosceles triangle are equal).

Since $\angle ABC$ is supplementary to $\angle CBE$, $\boxed{\angle ABC = 130°}$.

Since the opposite angles of a parallelogram are equal, $\boxed{\angle D = 130°}$.

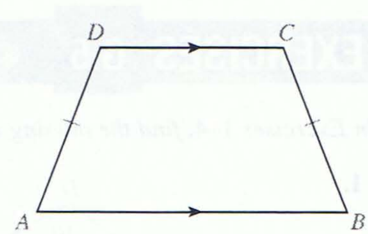
Now, since the sum of the angles of the parallelogram is 360°, the sum of the other two angles is $360° - 130° - 130° = 100°$. Therefore, $\boxed{\angle A = \angle BCD = 50°}$.

There is one more special type of quadrilateral that comes up frequently. If a quadrilateral has only one pair of parallel sides, it is called a ***trapezoid.*** The two parallel sides are called the ***bases*** of the trapezoid. In Figure G.51 the bases are labeled $b_1$ and $b_2$.




Trapezoid *ABCD*

**Figure G.51**




Isosceles trapezoid *ABCD*

**Figure G.52**

If the nonparallel sides of a trapezoid are of equal length, the trapezoid is called an ***isosceles trapezoid.*** Note that the base angles of an isosceles trapezoid are equal (think of extending sides $\overline{AD}$ and $\overline{BC}$ to form an isosceles triangle that would have the same base angles as the given isosceles trapezoid). See Figure G.52.

## EXAMPLE 5

Given the trapezoid *ABCD* in Figure G.53, find the length of the missing side.

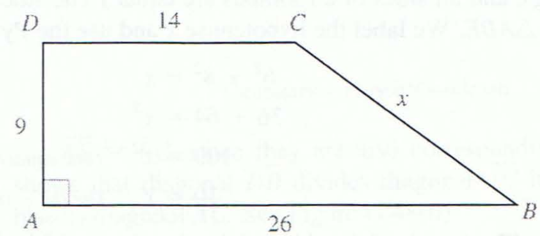



**Figure G.53**

**Solution** Let's draw a perpendicular line from vertex *C* to base *AB*, and label the intersection of this line segment and the bottom base *E* (see Figure G.54). We have divided trapezoid *ABCD* into rectangle *AECD* and right triangle *BEC*. First we focus on rectangle *AECD*. We can see $|\overline{EC}| = 9$, since it is the same length as the side of the rectangle opposite $\overline{EC}$; $|\overline{AE}| = 14$, since it is the same length as the side of the rectangle opposite $\overline{AE}$. Since $|\overline{AB}| = 26$, that leaves $|\overline{EB}| = 12$. Now we focus on the right triangle *EBC*. This triangle has legs 9 and 12. We can use the Pythagorean theorem to find the length of the missing side, which in this case, is the hypotenuse of the right triangle.

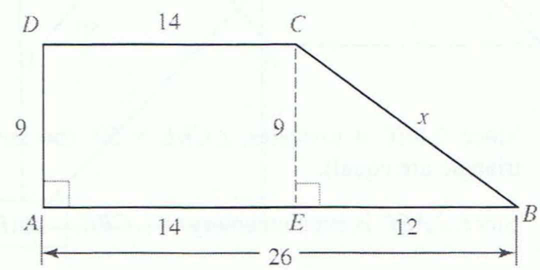



**Figure G.54**

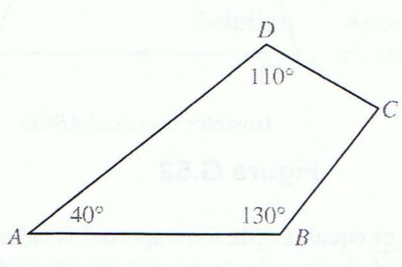
$$9^2 + 12^2 = x^2$$
$$81 + 144 = x^2$$
$$225 = x^2 \quad \textit{Take square roots to get}$$
$$15 = x \quad \textit{Use the positive root.}$$

The length of the missing side of the trapezoid is 15.

## EXERCISES G.5

*In Exercises 1–4, find the missing angles of the quadrilateral.*

**1.**




**2.**

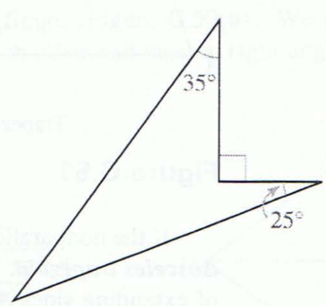
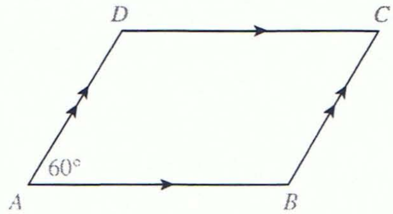
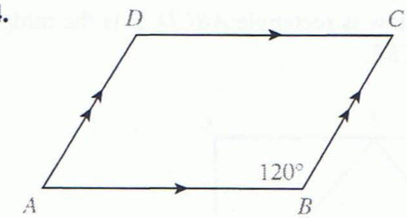



**Figure G.50** (a) (b)

**3.**

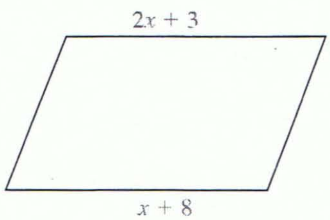



**4.**

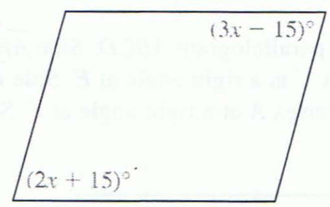



*In Exercises* 5–8, *find x. Each figure is a parallelogram.*

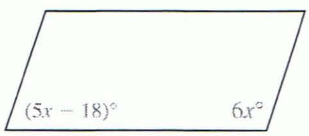
**5.**



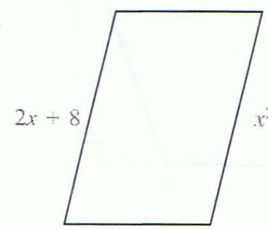

**6.**




**7.**




**8.**




**9.** Find the length of the diagonal of a rectangle whose base is 8 cm and whose height is 5 cm.

**10.** Find the length of the diagonal of a rectangle whose base is 6 cm and whose height is 9 cm.

**11.** Find the length of the diagonal of a square with side 4 in.

**12.** Find the length of the diagonal of a square with side 8 in.

**13.** Find the length of the side of a square with diagonal 5 cm.

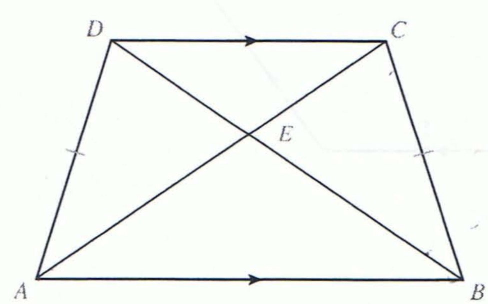
**14.** Find the length of the side of a square with diagonal 8 cm.

*The following figure is for Exercises* 15–17. *It is an isosceles trapezoid ABCD with* $\overline{AD} \cong \overline{CB}$.

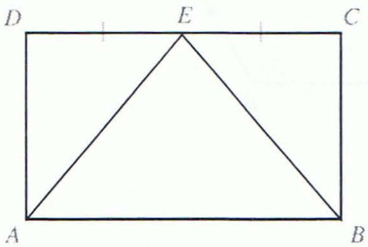
**15.** Referring to the figure at the right, show $\triangle ADB \cong \triangle BCA$.

**16.** Referring to the figure at the right, show $\triangle AEB \sim \triangle CED$.

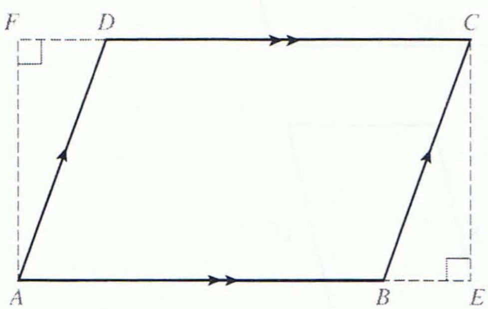
**17.** Referring to the figure at the right, if $\angle DCB \cong \angle CDA$, show $\triangle ADC \cong \triangle BCD$.



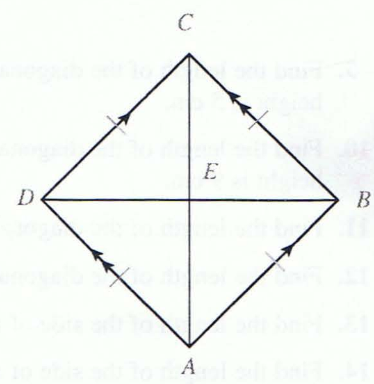
**18.** The figure below is rectangle $ABCD$. $E$ is the midpoint of $\overline{DC}$. Show $\angle DAE \cong \angle CBE$.




**19.** The figure below is parallelogram $ABCD$. Side $\overline{AB}$ is extended and meets the line segment from vertex $C$ at a right angle at $E$. Side $\overline{DC}$ is extended and meets the line segment from vertex $A$ at a right angle at $F$. Show $\triangle AFD \cong \triangle CEB$.




**20.** The figure to the right is rhombus $ABCD$ with diagonals $AC$ and $BD$ intersecting $E$. Note that the diagonals divide the rhombus into four triangles: $\triangle AEB$, $\triangle BEC$, $\triangle CED$, and $\triangle DEA$. Use the fact that the diagonals of a parallelogram bisect each other to show that all four triangles are congruent to each other. If all four triangles are congruent to each other, what can you conclude about the angles formed by the intersecting diagonals?

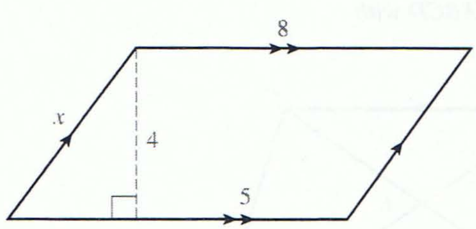



**21.** Find the length of the sides of a rhombus with diagonals 12″ and 18″.

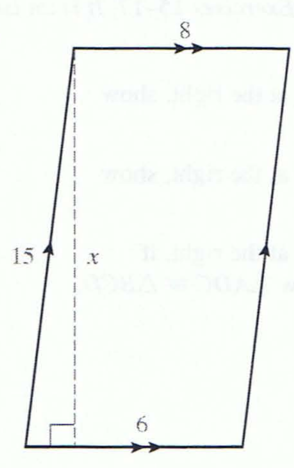
**22.** If the side of a rhombus is 10″ and a diagonal is 15″, find the length of the other diagonal to the nearest tenth.

*In Exercises* 23–30, *find x. Round to the nearest tenth where necessary.*

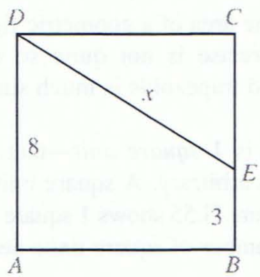
**23.**



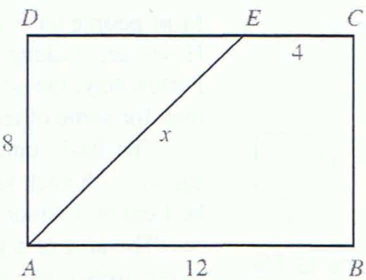

**24.**



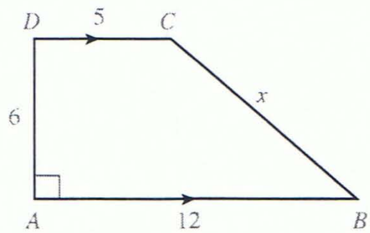
**25.** *ABCD* is a square.



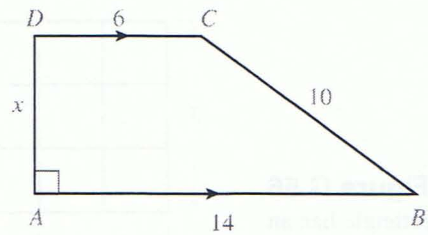

**26.** *ABCD* is a rectangle.



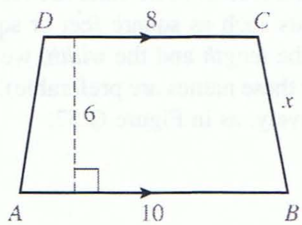

**27.** *ABCD* is a trapezoid.



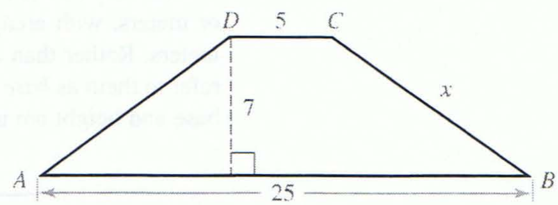

**28.** *ABCD* is a trapezoid.




**29.** *ABCD* is an isosceles trapezoid.




**30.** *ABCD* is an isosceles trapezoid.

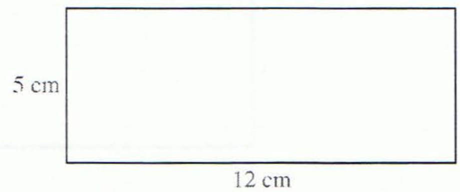



## G.6 Perimeter and Area

The perimeter of a polygon is simply the sum of the lengths of all its sides.

While there are a number of "formulas" for the perimeters of various geometric figures, they are just formal statements of this basic fact: To compute the perimeter of a polygon, we simply add the lengths of all its sides.

**EXAMPLE 1** Find the perimeter of the following rectangle.




**Solution** Since the opposite sides of a rectangle are equal, we have two sides of length 12 cm and two sides of length 5 cm. If we let $P =$ the perimeter, we have

$$P = 2(12) + 2(5) = 24 + 10 = \boxed{34 \text{ cm}}$$

## Area

Most people have an intuitive idea of what we mean by the area of a geometric figure. However, making our intuitive ideas mathematically precise is not quite so easy. Fortunately, the situation for parallelograms, triangles, and trapezoids is much simpler than for some other figures.

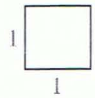



**Figure G.55**
1 square unit

The basic unit that we will use for measuring area is 1 *square unit*—that is, a square with each side one unit long. The unit of length is arbitrary. A square unit can be 1 cm by 1 cm or 1 foot by 1 foot or 1 mile by 1 mile. Figure G.55 shows 1 square unit.

The area of a geometric figure is defined to be the number of square units needed to precisely cover that figure.

If we consider a rectangle whose dimensions are 3 by 4 units (see Figure G.56), we can see that it contains 12 square units.

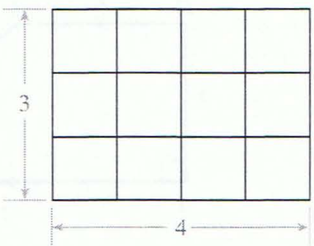



**Figure G.56**
A 3 by 4 rectangle has an area of 12 square units.

Be sure that you do not confuse length, which is measured in basic units such as feet or meters, with area, which is measured in square units such as square feet or square meters. Rather than calling the sides of the rectangle the *length* and the *width,* we will refer to them as *base* and *height* (you will soon see why these names are preferable). The base and height are usually labeled as $b$ and $h$, respectively, as in Figure G.57.

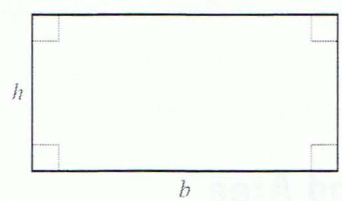
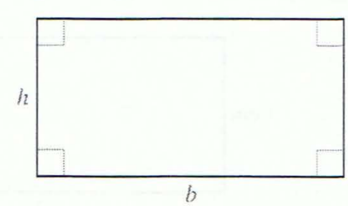



**Figure G.57**
Base and height of a rectangle

On the basis of our analysis of the case of the 3 by 4 rectangle, which clearly has an area of 12 square units, we can generalize and say that the area of a rectangle is the product of its base and height.

The area of a rectangle is given by $A = bh$.




Let us now consider a parallelogram $ABCD$ and let $b =$ the length of side $\overline{AB}$. See Figure G.58. We draw a perpendicular from $D$ to side $\overline{AB}$ and also from $C$ to the extension of side $\overline{AB}$ at $F$. Such a perpendicular line from a point to a line is often called an *altitude.* Since they are of equal length, we label both of these altitudes as $h$. All of this is shown in Figure G.58.

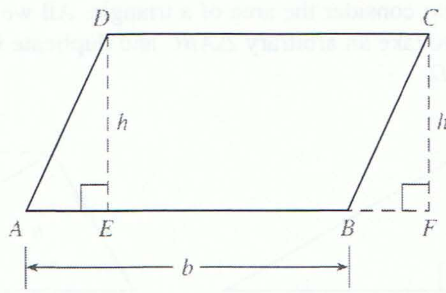
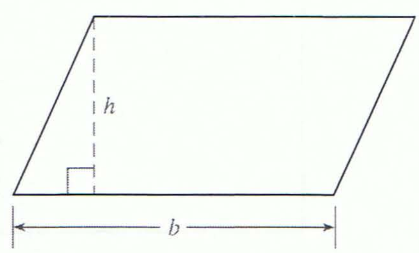



**Figure G.58**

Looking very carefully at Figure G.58, we can see several things. First, we see that $EFCD$ is a rectangle. Second, we can see that $\triangle ADE$ is congruent to $\triangle BCF$. We can think of cutting off $\triangle ADE$ from the left side of the parallelogram and pasting it on the right side as $\triangle BCF$. In this way we change the figure from a parallelogram into a rectangle but we *do not* change the area. Third, the length of $\overline{AB}$ is $b$, which is also the length of $\overline{EF}$. Therefore, the area of parallelogram $ABCD$ is equal to the area of rectangle $EFCD$, which is $bh$. We have thus established the following:

The area of a parallelogram is given by $A = bh$.

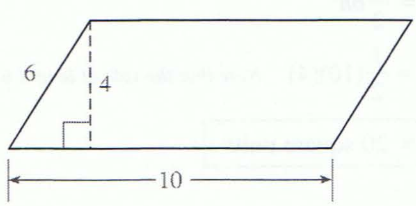



Be sure to recognize that for a rectangle the base and height are the two sides of the rectangle, whereas for a parallelogram the base is one side of the parallelogram but the height is the altitude drawn to that base from the opposite side.

Note that once a side is chosen as the base, $b$, the altitude to that base is the same length no matter whether it falls within the parallelogram or outside it, as can be seen in Figure G.58.

## EXAMPLE 2

Find the area of the following parallelogram.




**Solution** Using the formula for the area of a parallelogram, we have

$$A = bh$$
$$A = (10)(4)$$
$$\boxed{A = 40 \text{ square units}}$$

Note that we did *not* need to know the fact that the length of the other side of the parallelogram is 6.

We next consider the area of a triangle. All we need to do is look at Figure G.59, in which we take an arbitrary $\triangle ABC$ and duplicate it (as $\triangle BCD$) to produce parallelogram $ABCD$.

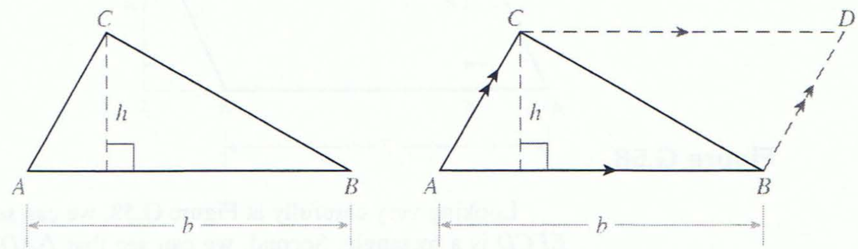
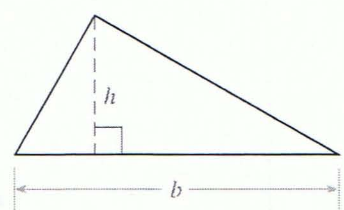



**Figure G.59**

Since $\triangle ABC$ is congruent to $\triangle BCD$, their areas are identical. Hence, the area of the triangle is one-half the area of the parallelogram, and so we have the following formula.

> The area of a triangle is given by $A = \frac{1}{2}bh$.
>
> 
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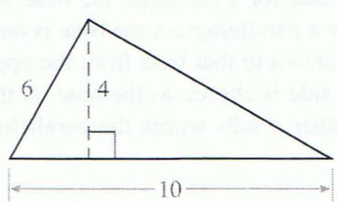
## EXAMPLE 3

Find the area of the following triangle.




**Solution**

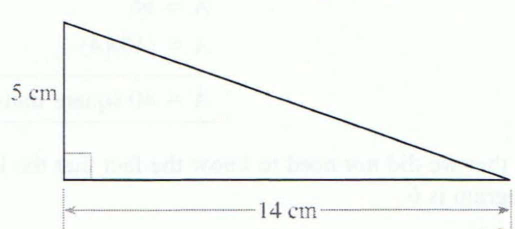
$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(10)(4)$$ *Note that the side of length 6 was unneeded information.*

$$\boxed{A = 20 \text{ square units}}$$

## EXAMPLE 4

Find the area of the following triangle.



**Solution** Finding the area of a *right* triangle can be particularly easy if we know the lengths of both legs. In such a case, we can use one of the legs as the base and the other leg as the height.

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(14)(5)$$

$$\boxed{A = 35 \text{ sq cm}}$$

## Perimeter and Area of Scaled Figures

In Section G.4 we discussed the idea of similarity. Let's examine the perimeter and area of similar triangles and quadrilaterals.

Consider the following two similar triangles in Figure G.60.

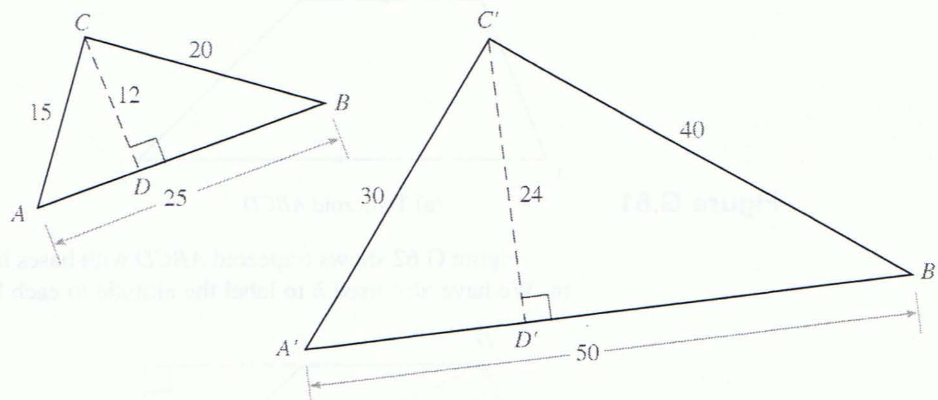



**Figure G.60**

The ratio of the corresponding sides of $\triangle ABC$ to those of $\triangle A'B'C'$ is $\frac{1}{2}$. The perimeter of $\triangle ABC$ is 60, and the perimeter of $\triangle A'B'C'$ is 120, so the ratio of the perimeters is also $\frac{1}{2}$. Of course, this should come as no surprise, since each side of $\triangle A'B'C'$ is twice the length of the corresponding side of $\triangle ABC$.

In effect, we have demonstrated that the perimeters of similar figures are in the same ratio as their sides. Alternatively, we can say that the same scaling factor that applies to the sides of the similar triangles also applies to the perimeters.

Let's now look at the area of these two similar triangles. Returning to Figure G.60, we have drawn in the heights $\overline{CD}$ and $\overline{C'D'}$ of the two triangles. We note that $\triangle ABC$ and $\triangle A'B'C'$ are also similar; therefore, the two heights are in the same ratio as the sides we have.

The area of $\triangle ABC = \frac{1}{2}(25)(12) = 150$

The area of $\triangle A'B'C' = \frac{1}{2}(50)(24) = 600$ *Note that* $600 = 4 \cdot 150$.

Since the base and height of $\triangle A'B'C'$ are *each* two times the base and height of $\triangle ABC$, the area of $\triangle A'B'C'$ is four times the area of $\triangle ABC$.

In general, if the scaling factor for two similar triangles is $K$, then both the base and the height are getting multiplied by $K$, and so the scaling factor for the *area* is $K^2$.

> If the scaling factor for two similar triangles is $K$, then the scaling factor for their *perimeter* is also $K$ and the scaling factor for their *area* is $K^2$.

**EXAMPLE 5** Suppose a side of one triangle is 20 in. and the corresponding side of a second similar triangle is 50 in. If the perimeter of the first triangle is 100 in. and its area is 600 sq in., find the perimeter and area of the similar triangle.

**Solution** From the ratio of the similar sides we determine that the scaling factor is $\frac{50}{20} = \frac{5}{2} = 2.5$.

Thus, to obtain the perimeter of the second triangle, we multiply by 2.5 to obtain $2.5(100) = \boxed{250 \text{ in.}}$.

To obtain the area of the second triangle, we multiply by $(2.5)^2 = 6.25$ to obtain $6.25(600) = \boxed{3{,}750 \text{ sq in.}}$.

The final figure we consider in this section is the trapezoid. Recall that a trapezoid is a quadrilateral with one pair of opposite sides parallel. (See Figure G.61(**a**).) If the nonparallel sides of a trapezoid are of equal length, the trapezoid is called an ***isosceles trapezoid.*** (See Figure G.61(**b**).)

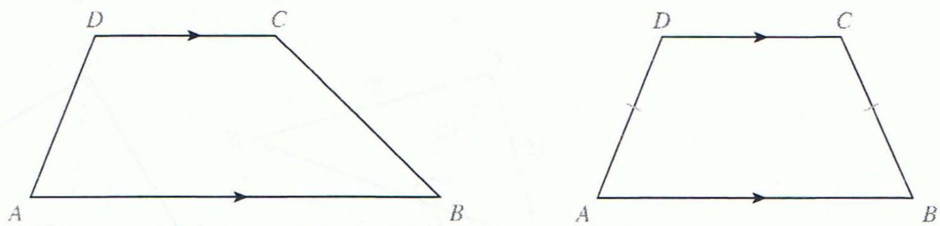



**Figure G.61** (**a**) Trapezoid *ABCD* (**b**) Isosceles trapezoid *ABCD*

Figure G.62 shows trapezoid *ABCD* with bases $b_1$ and $b_2$, with diagonal $\overline{BD}$ drawn in. We have also used $h$ to label the altitude to each base.

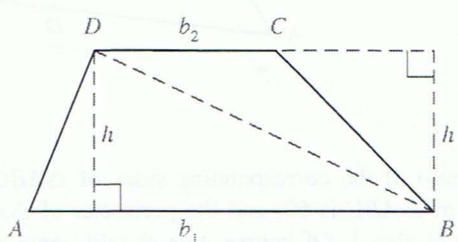



**Figure G.62**
Trapezoid *ABCD*

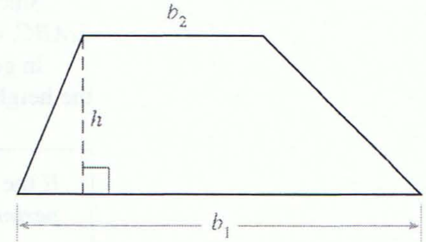
We note that the diagonal divides the trapezoid into two triangles, $\triangle ABD$ and $\triangle BCD$. The area of each triangle is one-half the base times the height. We thus have the following:

$$\text{Area of trapezoid } ABCD = Area\ \triangle ABD + Area\ of\ \triangle BCD$$
$$= \tfrac{1}{2}b_1h \quad + \tfrac{1}{2}b_2h$$
$$= \tfrac{1}{2}h(b_1 + b_2)$$

*Factor out the common factor of $\frac{1}{2}h$.*

We have thus derived the following:

The area of a trapezoid is given by
$A = \frac{1}{2}h(b_1 + b_2)$.



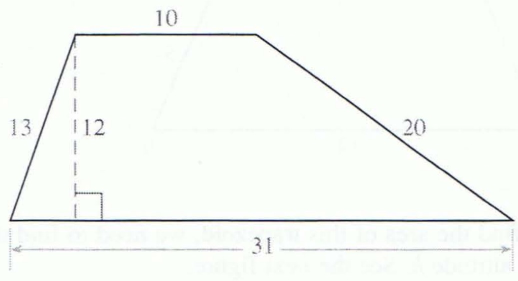
It is interesting to note that this formula can also be written as

$$A = h\left(\frac{b_1 + b_2}{2}\right)$$

When written this way, the formula is saying "the area of a trapezoid is the height times the *average* of the two bases."

## EXAMPLE 6

Find the perimeter and area of the following figure. (The units are inches.)




**Solution** The perimeter, $P$, is the sum of the four sides of the figure:

$$P = 31 + 10 + 20 + 13$$
$$P = 74 \text{ in.}$$

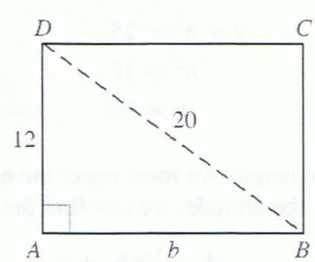
To find the area of the trapezoid, we use the formula

$$A = \tfrac{1}{2}h(b_1 + b_2)$$
$$A = \tfrac{1}{2}(12)(31 + 10)$$
$$A = (6)(41)$$
$$\boxed{A = 246 \text{ sq in.}}$$

## EXAMPLE 7

Find the area of a rectangle whose height is of length 12 in. and whose diagonal is of length 20 in.

**Solution** We draw the accompanying diagram. To find the area of the rectangle, we would like to find the length of the base, which we have labeled $b$. We note that $\triangle ABD$ is a right triangle, so we can apply the Pythagorean theorem.



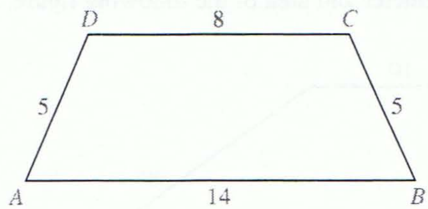

$$b^2 + 12^2 = 20^2$$
$$b^2 + 144 = 400$$
$$b^2 = 256 \quad \textit{Take square roots.}$$
$$b = \pm\sqrt{256} \quad \textit{Since the base must be positive, we have}$$
$$b = 16 \text{ in.}$$

Now that we know that the base is 16 in., we can find the area of the rectangle.

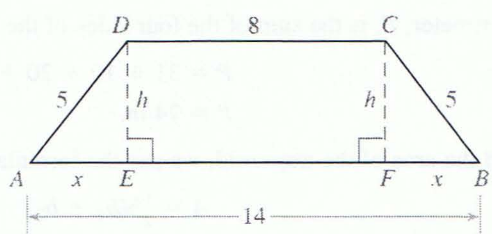
$$A = bh = (16)(12) = \boxed{192 \text{ sq in.}}$$

## EXAMPLE 8

Find the area of the isosceles trapezoid. (The units are meters.)




**Solution** To find the area of this trapezoid, we need to find the length of altitude $DE$. Let's call this altitude $h$. See the next figure.

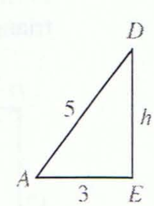



Since this is an isosceles trapezoid, the base angles are equal and so $\triangle ADE$ and $\triangle BCF$ are congruent. Thus, $\overline{AE}$ and $\overline{BF}$ are of equal length. Let's call this length $x$. Since the length of $\overline{AB}$ is 14 and $|\overline{EF}| = |\overline{DC}| = 8$, we have

$$\begin{aligned} x + x + 8 &= 14 \\ 2x + 8 &= 14 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Now we can look at $\triangle AED$; since it is a right triangle, we can apply the Pythagorean theorem.

$$\begin{aligned} 3^2 + h^2 &= 5^2 \\ 9 + h^2 &= 25 \\ h^2 &= 16 \\ h &= \pm 4 \end{aligned}$$




Since $h$ represents a length, we must reject the negative answer, so we have $h = 4$. Now that we know the altitude, we can find the area of trapezoid $ABCD$.

$$\begin{aligned} A &= \tfrac{1}{2}h(b_1 + b_2) \\ A &= \tfrac{1}{2}(4)(14 + 8) \\ A &= 2(22) \end{aligned}$$

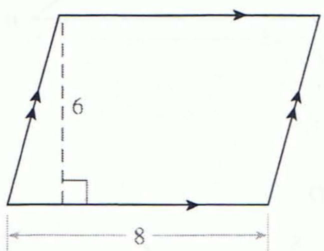
$$\boxed{A = 44 \text{ square meters}}$$

## EXERCISES G.6

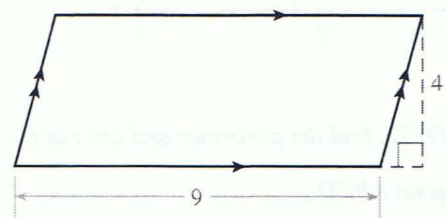
1. Find the perimeter and area of a rectangle whose width is 5 ft and whose length is 8 ft.
2. Find the perimeter and area of a rectangle whose width is 7 meters and whose length is 9 meters.
3. Find the perimeter and area of a rectangle whose dimensions are 6 in. by 12 in.
4. Find the perimeter and area of a rectangle with base 9 cm and height 8 cm.
5. Find the perimeter and area of a square with side 3 in.
6. Find the perimeter and area of a square with side 8 ft.

*In Exercises* 7–18, *find the area of the given figure.*

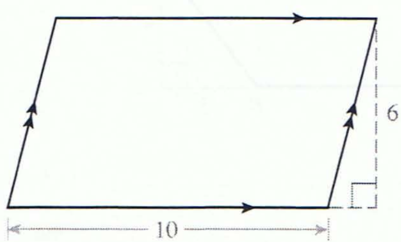
**7.**



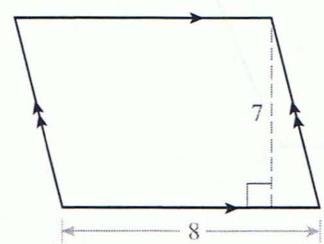

**8.**



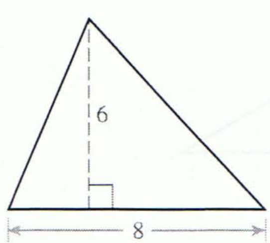

**9.**



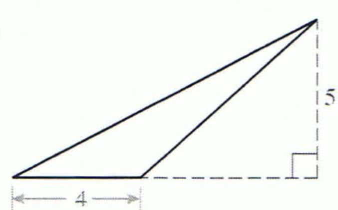

**10.**




**11.**




**12.**

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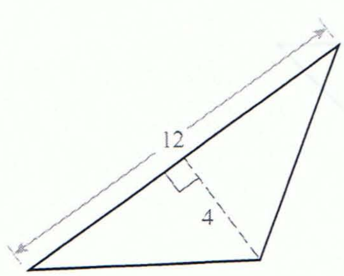


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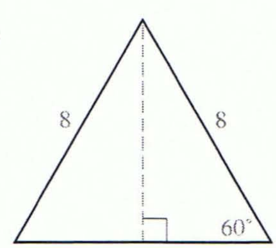
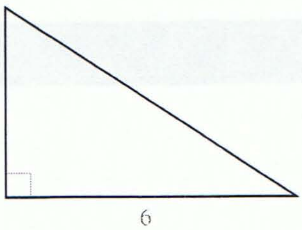
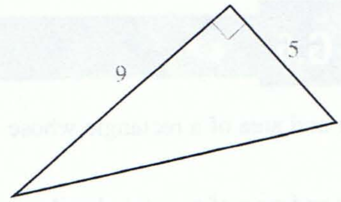


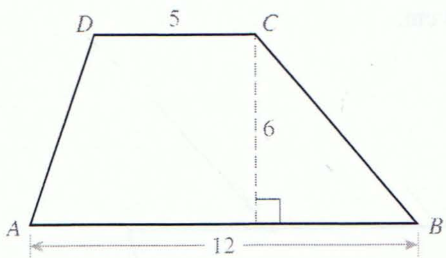
**15.**



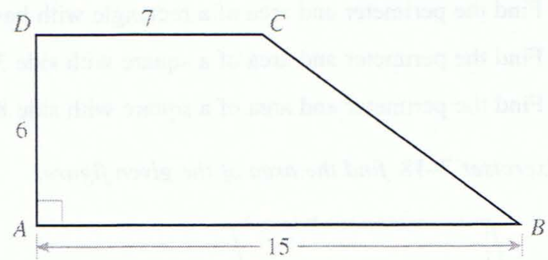

**16.**




**17.** Trapezoid $ABCD$

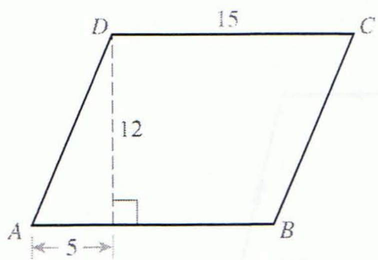



**18.** Trapezoid $ABCD$

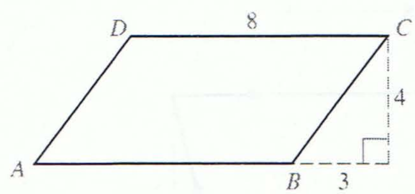



*In Exercises* 19–32, *find the perimeter and area of the given figure.*

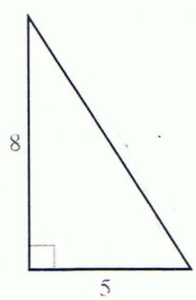
**19.** Parallelogram $ABCD$



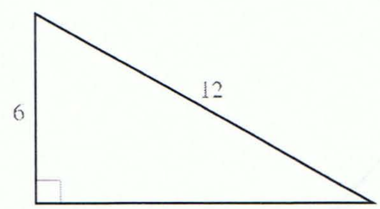

**20.** Parallelogram $ABCD$



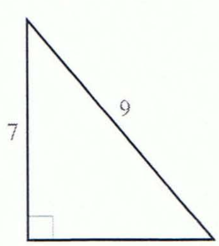

**21.**



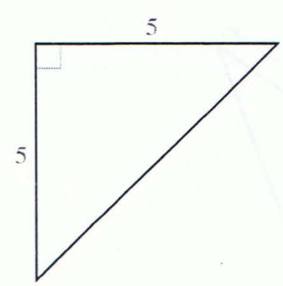

**22.**




**23.**




**24.**



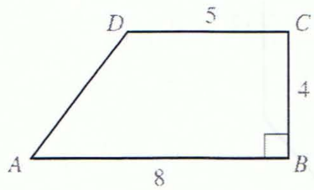
**25.** A rectangle with base 8 cm and diagonal 12 cm

**26.** A rectangle with height 5 in. and diagonal 9 in.

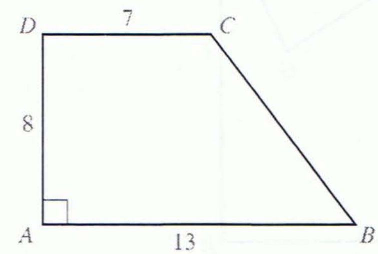
**27.** A square with diagonal 10 mm

**28.** A square with diagonal 15 meters

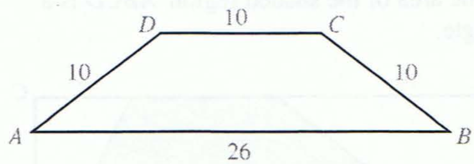
**29.** Trapezoid *ABCD*



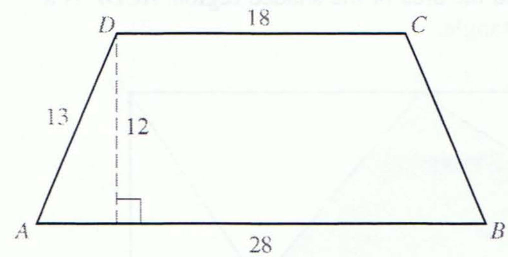

**30.** Trapezoid *ABCD*



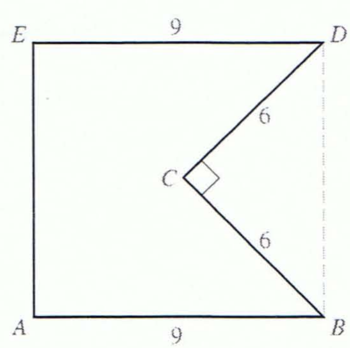

**31.** Isosceles trapezoid *ABCD*



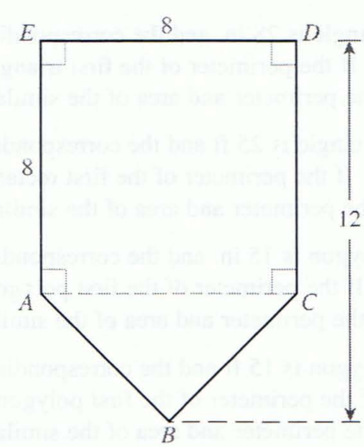

**32.** Isosceles trapezoid *ABCD*



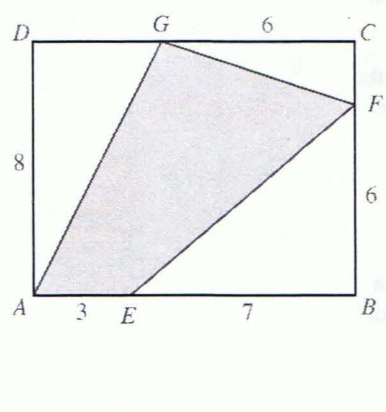

**33.** Find the area of *ABCDE*. *ABDE* is a rectangle.



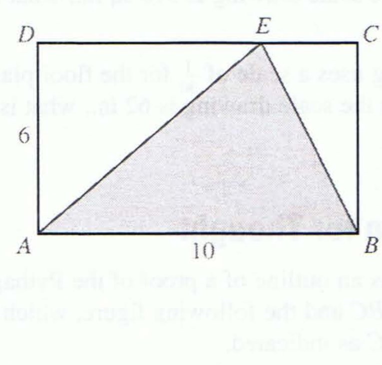

**34.** Find the area of *ABCDE*.



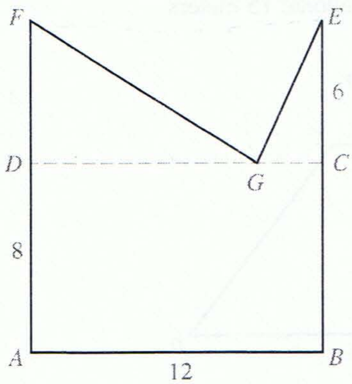

**35.** Find the area of *AEFG*. *ABCD* is a rectangle.



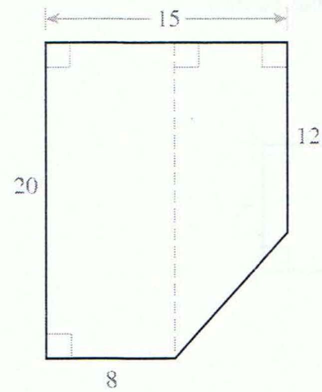

**36.** Find the area of $\triangle ABE$. *ABCD* is a rectangle.



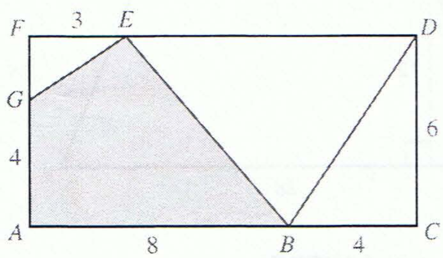
**37.** $|\overline{AF}| = |\overline{BE}|$. Find the area of the figure. $ABCD$ is a rectangle.



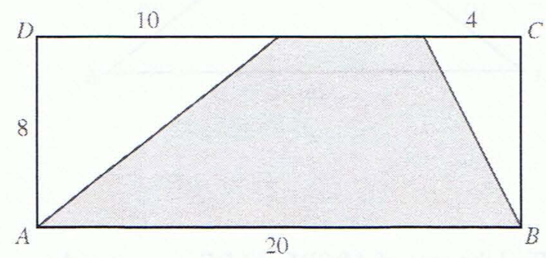

**38.** Find the area of the figure.




**39.** Find the area of the shaded region. $ACDF$ is a rectangle.




**40.** Find the area of the shaded region. $ABCD$ is a rectangle.




**41.** A side of one triangle is 28 in. and the corresponding side of a second similar triangle is 42 in.. If the perimeter of the first triangle is 98 in. and its area is 420 sq in., find the perimeter and area of the similar triangle.

**42.** A side of one rectangle is 25 ft and the corresponding side of a second similar rectangle is 10 ft. If the perimeter of the first rectangle is 150 ft and its area is 1250 sq ft, find the perimeter and area of the similar rectangle.

**43.** A side of one polygon is 15 in. and the corresponding side of a second similar polygon is 24 ft. If the perimeter of the first polygon is 80 in. and its area is 2000 sq in., find the perimeter and area of the similar polygon.

**44.** A side of one polygon is 15 ft and the corresponding side of a second similar polygon is 2 ft. If the perimeter of the first polygon is 80 ft and its area is 2000 sq ft, find the perimeter and area of the similar polygon.

**45.** A scale drawing uses a scale of $\frac{1}{20}$ for the floor plan of a house. If the area of the first floor on the scale drawing is 378 sq in., what is the actual area of the first floor in sq ft?

**46.** A scale drawing uses a scale of $\frac{1}{30}$ for the floor plan of a house. If the perimeter of the house on the scale drawing is 62 in., what is the actual perimeter of the house in ft?

## ? Question for Thought

**47.** The following is an outline of a proof of the Pythagorean theorem. Consider a right triangle $ABC$ and the following figure, which is created by arranging four copies of $\triangle ABC$ as indicated.

Justify and/or explain each of the following statements.

**(a)** The entire figure and the figure in the middle are squares.

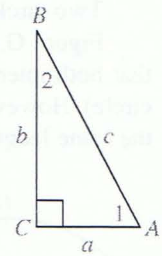
**(b)** The area of the outer square is $(a + b)^2$.

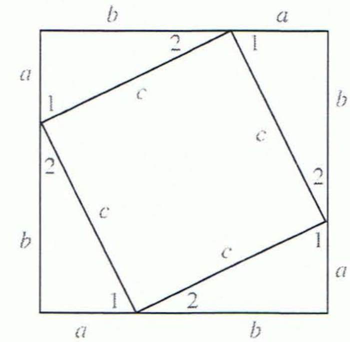
**(c)** The area of the inner square is $c^2$.

**(d)** The area of each triangle is $\frac{1}{2}ab$.

**(e)** The area of the outer square is equal to the area of the inner square plus the area of the four inner triangles.

**(f)** Use these results to prove the Pythagorean theorem.




## G.7 Circles

A ***circle*** is a set of points all of which are the same distance from a given point. The given point is called the ***center*** of the circle and is usually denoted by the letter $O$. The common distance from all points on the circle to the center is called the ***radius*** (plural *radii*). Figure G.63 shows circle $O$ with radius $r$.

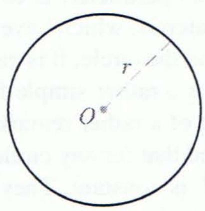



**Figure G.63**
Circle $O$ with radius $r$

A line segment from one point on a circle to another is called a ***chord.***

A chord that passes through the center of the circle is called a ***diameter.***

A ***secant*** is a line (or line segment) that passes through two points on the circle.

A ***tangent*** is a line (or line segment) outside a circle that intersects the circle in exactly one point.

Figure G.64 shows a circle with diameter $\overline{AB}$, chord $\overline{CD}$, secant $S$, and tangent $T$ drawn in.

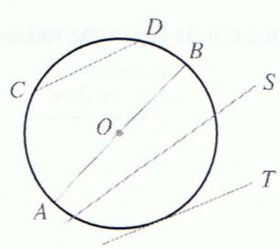



**Figure G.64**

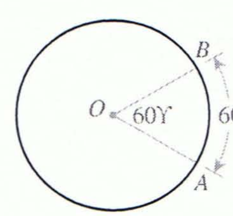



**Figure G.65**

A ***central angle*** of a circle is an angle whose vertex is at the center of the circle.

An ***arc*** of a circle is a portion of the circle that lies between two points on the circle. The arc of a circle connecting the points $A$ and $B$ is denoted by $\widehat{AB}$.

In Figure G.65 we have central angle $BOA$ intercepting $\widehat{AB}$. The degree measure of an arc is defined to be the degree measure of its central angle. Thus, central angle $AOB$ of 60° will intercept $\widehat{AB}$ of 60 degrees (since both are one sixth of the entire circle).

Two circles are called ***concentric*** if they have the same center.

Figure G.66 shows two concentric circles with central angles of 90°. We can see that both intercepted arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are 90° (they are both one quarter of the entire circle). However, while both arcs have the same number of degrees, they are clearly not the same length.

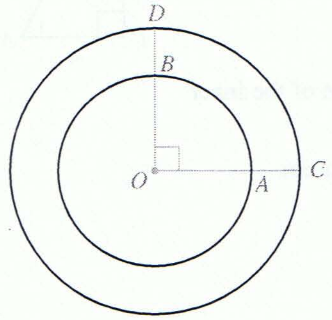



**Figure G.66**

Be careful not to confuse the *length* of an arc (which we have not yet discussed) with its degree measure.

## Circumference

The distance around a circle, or its perimeter, is called its ***circumference.*** Unlike the situation for triangles and quadrilaterals, which have sides to measure, we have no easy way to measure the distance around the circle. It is generally difficult to measure curved lines accurately. However, there *is* a rather simple formula for the circumference of a circle that is a direct consequence of a rather remarkable fact.

The ancient Greeks discovered that for any circle, large or small, the ratio of its circumference, $C$, to its diameter, $d$, is constant. They called this constant $\pi$ (the Greek letter pi).* Symbolically, we may write

$$\frac{C}{d} = \pi$$

If we multiply both sides of this equation by $d$, we obtain the following formula.

The circumference of a circle is given by

$$C = \pi d$$

Since the diameter of a circle is twice the radius, we may also write

$$\boxed{C = 2\pi r}$$

*The number $\pi$ is an irrational number, which means that its decimal representation never stops and never repeats. The great mathematician Archimedes is often credited with obtaining an early accurate estimate for the value of $\pi$. He estimated $\pi$ to be between $3\frac{1}{7}$ and $3\frac{10}{71}$. Today, $\pi$ has been computed to hundreds of thousands of decimal places. The value of $\pi$ accurate to five decimal places is 3.14159.

## EXAMPLE 1

Find the circumference of a circle whose diameter is 9 in. Give your answer in terms of $\pi$ and also to the nearest tenth.

**Solution** Using the formula for the circumference of a circle, we get

$$C = \pi d = \pi \cdot 9$$

$$\boxed{C = 9\pi \text{ in.}}$$

*Most scientific calculators have a* $\boxed{\pi}$ *key. Using this key, we could do this computation as* $\boxed{9}$ $\boxed{\times}$ $\boxed{\pi}$ $\boxed{=}$ *and the display will read* $\boxed{28.274334}$*, which we would round off to 28.3.*

If we want a numerical answer, we generally use an approximate value for $\pi$, $\pi \approx 3.14$. Thus,

$$C = \pi d = (3.14)(9)$$

$$C \approx 28.26 \quad \textit{Round off to the nearest tenth.}$$

$$\boxed{C = 28.3}$$

A ***sector*** of a circle is that portion of a circle enclosed by a central angle. A sector is the shaded portion illustrated in Figure G.67.

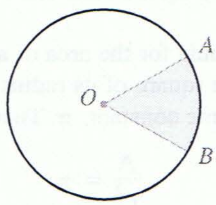


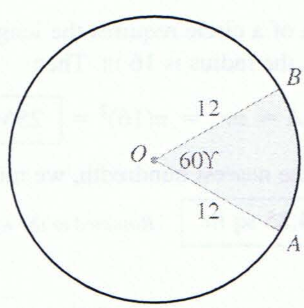

**Figure G.67**

## EXAMPLE 2

Given the following sector with a central angle of 60° in a circle whose radius is 12 cm, find the following:

**(a)** The length of $\widehat{AB}$ **(b)** The perimeter of sector $AOB$

**Solution** We have drawn a diagram containing the given information and shaded in the sector whose perimeter we are trying to find. The perimeter of the sector is equal to the sum of the lengths of the two radii plus the length of the arc $\widehat{AB}$.




**(a)** Since the central angle of the sector is 60°, which is one sixth of 360°, the length of arc $\widehat{AB}$ is going to be $\frac{1}{6}$ of the circumference of the entire circle. Therefore, the length of arc $\widehat{AB}$ is

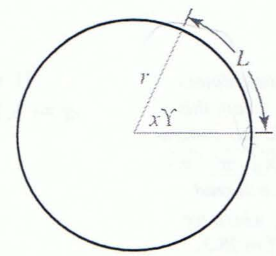
$$\frac{1}{6}(\text{Circumference of entire circle}) = \frac{1}{6}\pi d = \frac{1}{6}(\pi)24 = \boxed{4\pi \text{ cm}}$$

**(b)** The perimeter of the sector is $12 + 12 + 4\pi = \boxed{24 + 4\pi \text{ cm}}$

The *length* of arc $\overset{\frown}{AB}$ is actually a fractional portion of the circumference of the circle. That fraction is determined by the size of the central angle, $x$. Since the circumference of a circle is $2\pi r$ and a circle is 360°, we have the following.

The lenth, $L$, of an arc with central angle $x°$ is

$$L = \frac{x}{360}(2\pi r)$$




## Area

Finding the area of a circle presents an even greater problem than finding its perimeter. How are we to find out how many square units can be fitted into a region with a curved boundary?

Fortunately, we have a formula for the area of a circle. The Greeks found that the ratio of the area of a circle to the square of its radius is also constant. The really amazing fact is that the ratio is the same constant, $\pi$. That is,

$$\frac{A}{r^2} = \pi$$

Thus, we have the following formula.

The area of a circle is given by

$$A = \pi r^2$$

**EXAMPLE 3** Find the area of a circle whose diameter is 32 in. Give your answer in terms of $\pi$ and to the nearest hundredth.

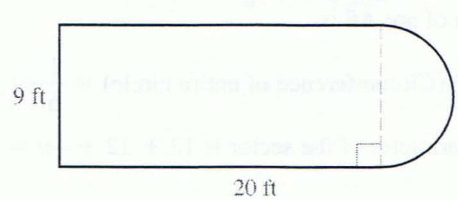
**Solution** The formula for the area of a circle requires the length of the radius. Since the diameter of the circle is 32 in., the radius is 16 in. Then

$$A = \pi r^2 = \pi(16)^2 = \boxed{256\pi \text{ sq in.}}$$

To compute the area to the nearest hundredth, we multiply $256\pi$ and get

$\boxed{804.25 \text{ sq in.}}$ *Rounded to the nearest hundredth*

**EXAMPLE 4** Find the area of the following figure, composed of a rectangle and a semicircle. Round off your answer to the nearest square foot.



**Solution** The dashed line is not actually part of the figure; it is drawn in just to make the figure clearer. The total area of this figure, $A$, is the area of the rectangle plus the area of the semicircle. To compute the area of the semicircle, we need to know its radius. Since the opposite sides of a rectangle are equal, the diameter of the semicircle is 9, and its radius is $\frac{9}{2}$. Therefore, we have

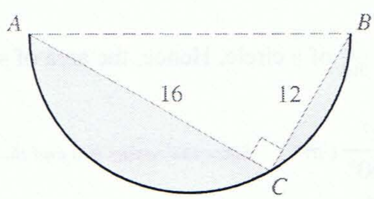
$$\begin{aligned} A &= A_{\text{rectangle}} + A_{\text{semicircle}} \\ &= bh + \frac{1}{2}\pi r^2 \\ &= (20)(9) + \frac{1}{2}\pi\left(\frac{9}{2}\right)^2 \\ &= 180 + \frac{1}{2}\pi\left(\frac{81}{4}\right) \end{aligned}$$

$$\boxed{A = 180 + \frac{81\pi}{8} \text{ sq ft}}$$ *Using the approximate value 3.14 for $\pi$ and rounding to the nearest sq ft, we have*

$$\boxed{A \approx 212 \text{ sq ft}}$$

## EXAMPLE 5

Find the area of the shaded portion of the following figure. Arc $ACB$ is a semicircle.




**Solution** We can visualize the required area as the area of the semicircle minus the area of the right triangle $ABC$. To get the area of the semicircle, we need to find its radius. $\overline{AB}$ is the diameter of the semicircle and the hypotenuse of right triangle $ABC$. Therefore, we can find the length $\overline{AB}$ by using the Pythagorean theorem.

$$|\overline{AB}|^2 = 12^2 + 16^2$$

$$|\overline{AB}|^2 = 144 + 256 = 400 \quad \textit{Take square roots.}$$

$$|\overline{AB}| = 20 \quad \textit{Since a length must be positive}$$

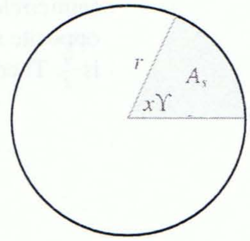
Now we can compute the required area.

$$\begin{aligned} \text{Area of shaded portion of figure} &= \text{Area of semicircle} - \text{Area of triangle} \\ &= \frac{1}{2}\pi r^2 - \frac{1}{2}bh \quad \textit{Since the diameter is 20, the radius is 10.} \\ &= \frac{1}{2}\pi(10)^2 - \frac{1}{2}(12)(16) \\ &= \boxed{50\pi - 96 \text{ square units}} \end{aligned}$$

The area of a *sector* is a fractional part of the area of the circle. That fraction is determined by the size of the central angle, $x$. Since the area of a circle is $\pi r^2$ and a circle is 360°, we have the following:

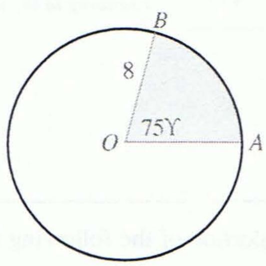
The area of a sector, $A_s$, with central $x°$ is

$$A_s = \frac{x}{360}(\pi r^2)$$




## EXAMPLE 6

To the nearest hundredth of a centimeter, find the area of a sector with central angle 75° and radius 8 cm. (See the figure below.)




**Solution** A sector of 75° is $\frac{75}{360}$ of a circle. Hence, the area of sector $AOB$ is $\frac{75}{360}$ of the area of the circle.

$$A_s = \frac{x°}{360°}(\pi r^2)$$ *Since the radius is 8 and the central angle is 75°, we have*

$$A_s = \frac{75}{360}\pi(8^2)$$

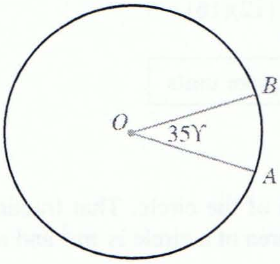
$$A_s = \frac{75\pi(64)}{360}$$ *This simplifies to*

$$A_s = \frac{40\pi}{3} \approx \boxed{41.89 \text{ cm}^2}$$

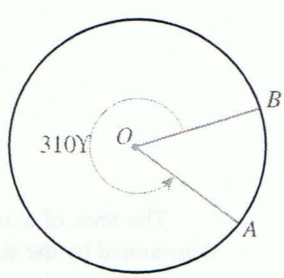
## EXERCISES G.7

*In Exercises 1–4, find the number of degrees in $\widehat{AB}$.*

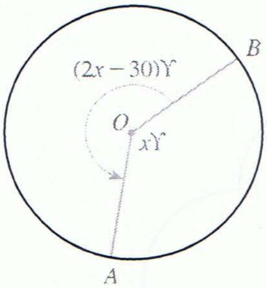
**1.**



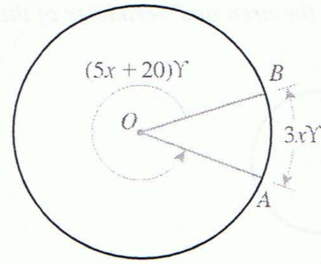

**2.**



**3.**




**4.**




**5.** Find the circumference of a circle with diameter 6 in.

**6.** Find the circumference of a circle with diameter 8 cm.

**7.** Find the circumference of a circle with radius 4 ft.

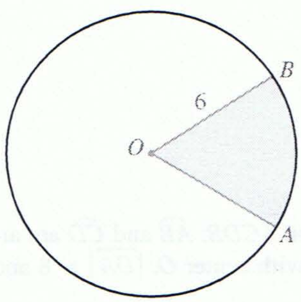
**8.** Find the circumference of a circle with radius 10 mm.

**9.** Find the arc length of a 30° sector of a circle with a radius of 9 in.

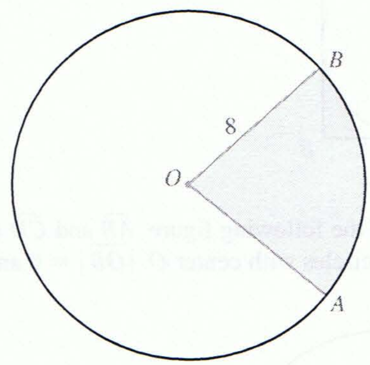
**10.** Find the arc length of a 60° sector of a circle with a radius of 9 in.

*In Exercises* 11–14, *find the perimeter and area of the indicated sector.*

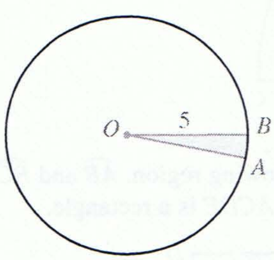
**11.** $\angle AOB = 65°$



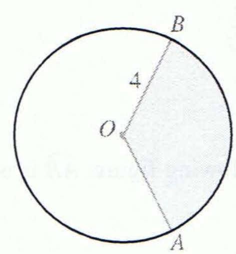

**12.** $\angle AOB = 80°$




**13.** $\angle AOB = 10°$




**14.** $\angle AOB = 120°$




**15.** Find the area of a circle with radius 3 in.

**16.** Find the area of a circle with radius 8 cm.

**17.** Find the area of a circle with diameter 12 ft.

**18.** Find the area of a circle with diameter 9 meters.

**19.** Find the perimeter and area of a semicircle with radius 10 in.

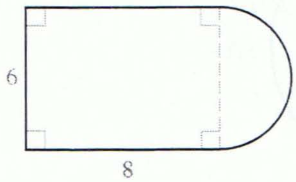
**20.** Find the perimeter and area of a semicircle with diameter 10 in.

**21.** Find the area of a sector with a central angle of 80° and a radius of 5 in.

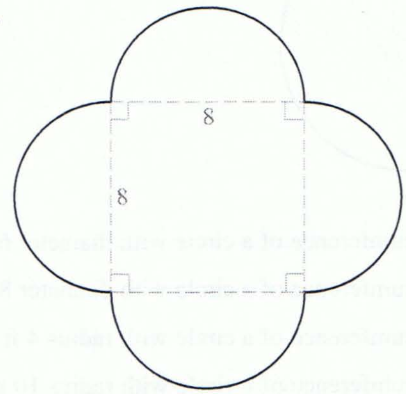
**22.** Find the area of a sector with a central angle of 40° and a radius of 8 cm.

*In Exercises 23–24, find the area and perimeter of the given figure. All the arcs are semicircles.*

**23.**

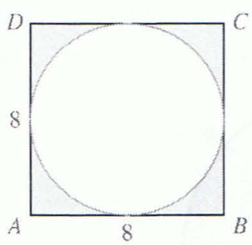



**24.**

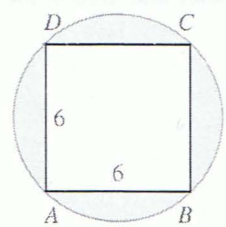



*In Exercises 25–26, find the area of the shaded region.*

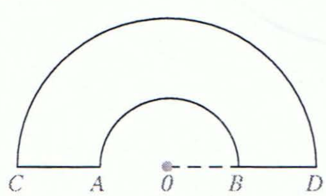
**25.** $ABCD$ is a square of side 8 cm.



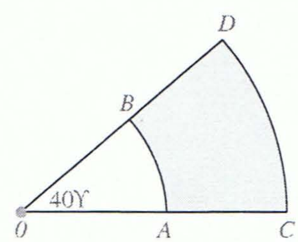

**26.** $ABCD$ is a square of side 6 cm.



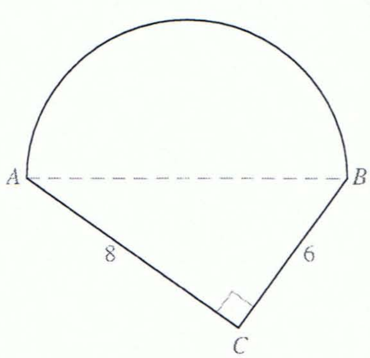

**27.** Find the area of the following figure. $\widehat{AB}$ and $\widehat{CD}$ are concentric semicircles with center $O$. $|\overline{OB}| = 6$ and $|\overline{BD}| = 8$.



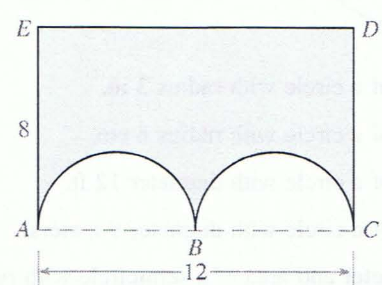

**28.** Find the area of region $ACDB$. $\widehat{AB}$ and $\widehat{CD}$ are arcs of concentric circles with center $O$. $|\overline{OA}| = 8$ and $|\overline{AC}| = 2$.



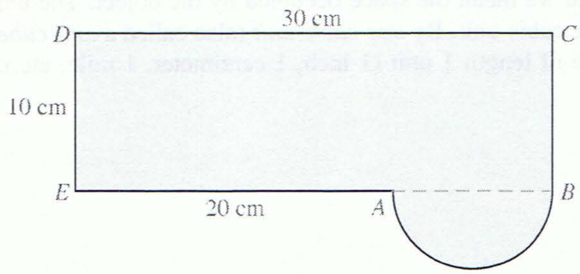

**29.** Find the perimeter of the following figure. $\widehat{AB}$ is a semicircle.



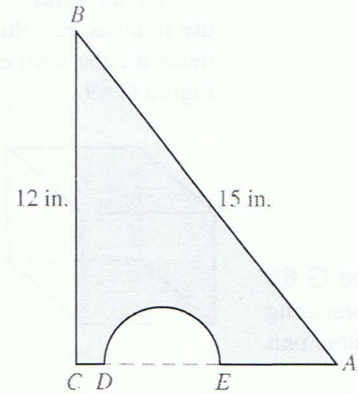

**30.** Find the area of the following region. $\widehat{AB}$ and $\widehat{BC}$ are congruent semircircles. $ACDE$ is a rectangle.



**31.** Find the perimeter and area of the given figure. $BCDE$ is a rectangle. $\widehat{AB}$ is a semicircle.




**32.** Find the area of the shaded region. $ABC$ is a right triangle, $\widehat{DE}$ is a semicircle of radius 2 in.




**33.** Which contains more pizza: one round 12-in. (diameter is 12 in.) pie or two round 8-in. pies? If a 12-in. pie costs \$9 and an 8-in. pie costs \$4, which is the better buy?

**34.** Repeat Exercise 33 for one 15-in. pie that costs \$12 or two 10-in. pies that cost \$8 each.

## G.8 Solid Geometry

Our study of solid geometry will deal with the surface area and volume of certain three-dimensional objects. By the ***surface area*** of an object we mean the area of the exterior of the object. For example, consider the solid box pictured in Figure G.68 with length $L$, width $W$, and height $H$. The values of $L$, $W$, and $H$ are called the *dimensions* of the rectangular box. The sides of the solid are called its *faces*, and because these faces are all rectangles, it is called a *rectangular solid*. The line segments forming the sides of the rectangles are called the *edges* of the solid.

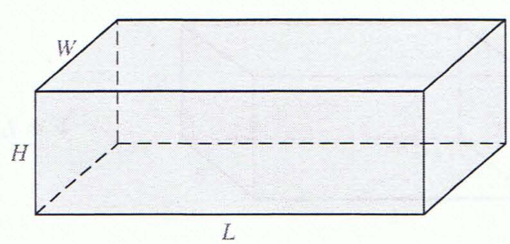
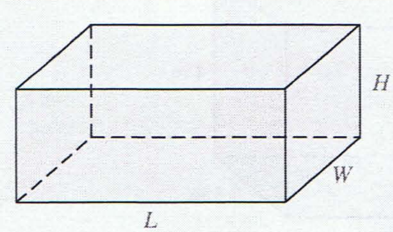



**Figure G.68**
A rectangular solid

The surface area of a rectangular solid is the sum of the areas of its six faces. Since the top and bottom faces are identical, the front and back faces are identical, and the left and right faces are identical, we can compute the surface area, $S$, of the rectangular solid by adding up the area of its six faces to obtain the following formula.

**The Surface Area of a Rectangular Solid**




$$S = 2LW + 2LH + 2HW$$

If the length, width, and height of a rectangular solid are all equal, the solid is called a ***cube.***

By the ***volume*** of a solid we mean the space occupied by the object. The unit we use to measure volume is the cubic unit. By *one cubic unit* (also called a *unit cube*) we mean a cube with each edge of length 1 unit (1 inch, 1 centimeter, 1 mile, etc.). See Figure G.69.

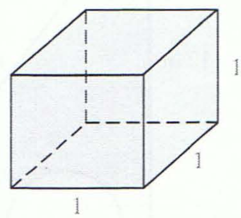



**Figure G.69**
The basic unit for measuring volume: One cubic unit

Measuring the volume of a solid means calculating how many cubic units it takes to fill up the solid. In the case of a rectangular solid, this is not very difficult. For example, Figure G.70 illustrates how a rectangular solid of dimensions 6 cm by 2 cm by 3 cm can be sliced up into 36 unit cubes, and so its volume is 36 cubic centimeters (often written 36 $cm^3$).

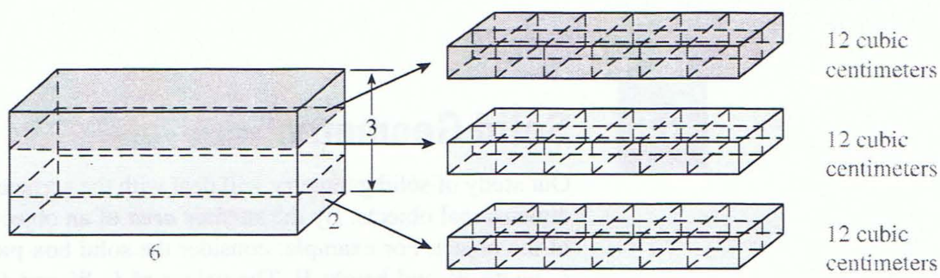
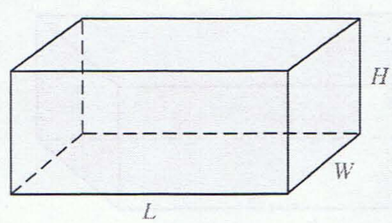



**Figure G.70**
Computing the volume of a rectangular solid

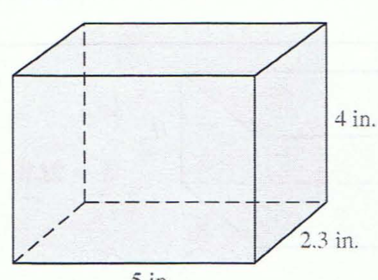
We can generalize the result of Figure G.70 to obtain the following formula.

**The Volume of a Rectangular Solid**




$$V = LWH$$

**EXAMPLE 1** Find the surface area and volume of the following rectangular box.



**Solution** The surface area is

$$\begin{aligned} S &= 2LW + 2LH + 2HW \\ &= 2(5)(2.3) + 2(5)(4) + 2(4)(2.3) \\ &= \boxed{81.4 \text{ sq in.}} \end{aligned}$$

*We substitute $L = 5$, $W = 2.3$, and $H = 4$.*

The volume is

$$\begin{aligned} V &= LWH \\ &= 5(2.3)(4) \\ &= \boxed{46 \text{ cubic in.}} \end{aligned}$$

## The Cylinder

Recall that to find the area of a circle with radius 3 in., we use the formula $A = \pi r^2$ to obtain $A = \pi(3)^2 = 9\pi \approx 28.3$ sq in. This means that approximately 28.3 unit squares are needed to cover a circle of radius 3. (See Figure G.71.)

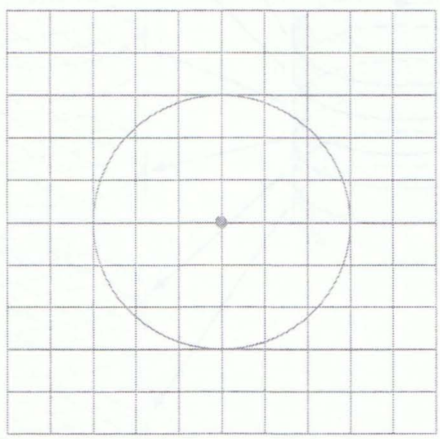


**Figure G.71**
A circle of radius 3 in. has an area of approximately 28.3 sq in.

A circle is a two-dimensional figure. The three-dimensional figure formed by moving the circle parallel to itself, as illustrated in Figure G.72, is called a ***right circular cylinder.***

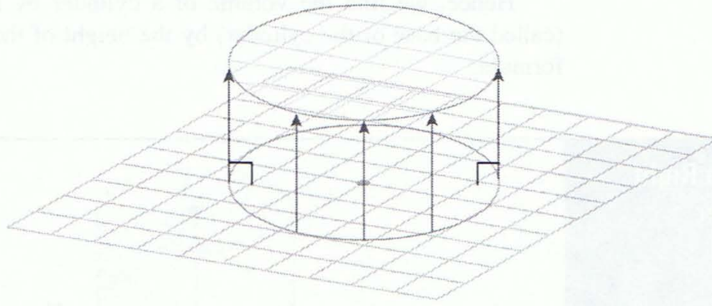


**Figure G.72**
Forming a right circular cylinder

As with a rectangular solid, finding the volume of a cylinder means determining the number of *cubic* units it takes to fill up the cylinder. Let's examine the cylinder formed by moving a circle of radius 3 in. parallel to itself to a height of 1 in. See Figure G.73.

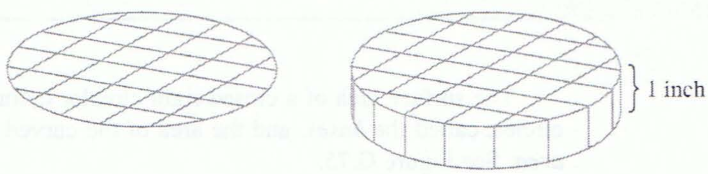
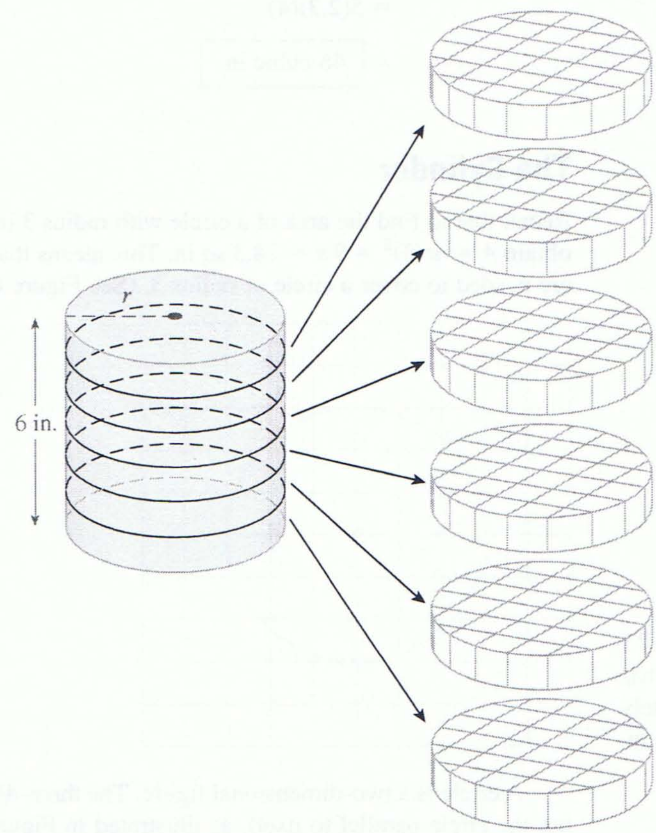



**Figure G.73**

As we can see from Figure G.73, each of the 28.3 sq in. in the area of the circle will generate a cubic in.. Hence, a circle of area 28.3 sq in. will generate a cylinder made up of 28.3 cubic in.

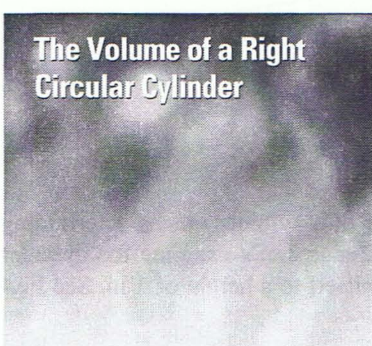
Similarly, Figure G.74 illustrates a cylinder generated by a circle of radius $r$ in. moved parallel to itself through a height of 6 in.. Examining the individual layers, we recognize that each of the $\pi r^2$ in. in the area of the circle will generate a cubic in.. The volume of each layer is $\pi r^2$ times a height of 1 in. Thus, the volume of this cylinder is $6(\pi r^2) = 6\pi r^2$ cubic in.



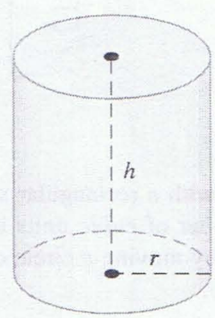

*The volume of each layer is $\pi r^2$ cubic inches.*

**Figure G.74**
The volume of a cylinder of radius $r$ in. and height 6 in.

Hence, we find the volume of a cylinder by multiplying the area of the circle (called the base of the cylinder) by the height of the cylinder. We have the following formula.



## The Volume of a Right Circular Cylinder




$$V = \pi r^2 h$$

The surface area of a closed right circular cylinder consists of the area of the two circles, called the ***bases,*** and the area of the curved surface, called the ***lateral surface area.*** See Figure G.75.

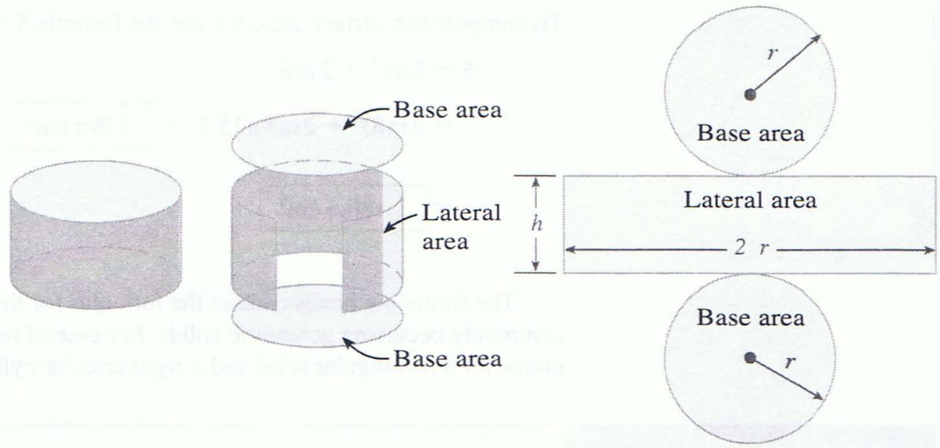



**Figure G.75**
The surface area of a closed right circular cylinder

As illustrated in Figure G.75, if we cut the cylinder open and lay it flat, we can find its surface area by adding the area of the two bases (which are circles) and the lateral area (which is a rectangle). The area of each circle is $\pi r^2$, where $r$ is the radius of the circle. To find the lateral area, the area of the rectangle, we note that the height of the rectangle is $h$, the height of the cylinder. Let's look at the width of the rectangle. Note that if we were to reconstruct the cylinder, the width of the rectangle wraps around the bases of the cylinder (which are circles). Therefore, the width of the rectangle is equal to the circumference of the circle, which is $2\pi r$. Hence,

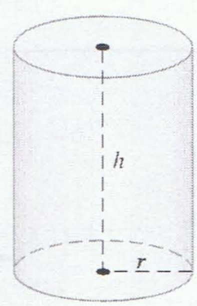
$$\text{Lateral surface area} = (\text{Base of rectangle})(\text{Height of rectangle}) = (2\pi r)(h)$$

We have shown that

$$\begin{array}{c} \text{Total surface area of} \\ \text{a right circular cylinder} \end{array} = \text{Area of the two bases} + \text{Lateral surface area}$$

$$S = 2\pi r^2 + 2\pi rh$$

## The Total Surface Area of a Closed Right Circular Cylinder




$$S = 2\pi r^2 + 2\pi rh$$

## EXAMPLE 2

Find the volume and surface area of a closed cylinder of radius 8 cm and height 15.5 cm. Give your answers in terms of $\pi$ and rounded to the nearest tenth.

**Solution** To compute the volume, we use the formula $V = \pi r^2 h$:

$$V = \pi r^2 h$$ *Substitute $r = 8$ and $h = 15.5$.*

$$= \pi(8)^2(15.5) = \boxed{992\pi \text{ cm}^3}$$ *Using $\pi = 3.14$ and rounding to the nearest tenth, we get*

$$= \boxed{3{,}114.9 \text{ cm}^3}$$

To compute the surface area, we use the formula $S = 2\pi r^2 + 2\pi rh$.

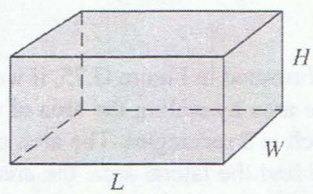
$$S = 2\pi r^2 + 2\pi rh \qquad \textit{Substitute } r = 8 \textit{ and } h = 15.5.$$

$$= 2\pi(8)^2 + 2\pi(8)(15.5) = \boxed{376\pi \text{ cm}^2} \qquad \textit{Using } \pi \approx 3.14 \textit{ and rounding to the nearest tenth, we get}$$

$$= \boxed{1{,}180.6 \text{ cm}^2}$$

The following boxes contain the formulas for the surface area and volume of some commonly occurring geometric solids. For ease of reference, we have included the formulas for a rectangular solid and a right circular cylinder.

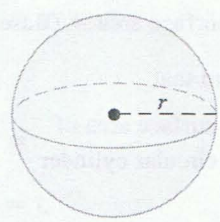
### Rectangular Solid




$S = 2LW + 2LH + 2HW$

$V = LWH$

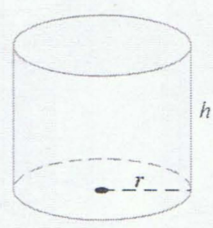
### Sphere




$S = 4\pi r^2$

$V = \dfrac{4}{3}\pi r^3$

### Right Circular Cylinder




*Lateral surface area*

$S = 2\pi rh$

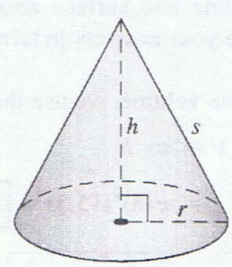
*Total surface area*

$S = 2\pi rh + 2\pi r^2$

*Volume*

$V = \pi r^2 h$

### Right Circular Cone




*Lateral surface area*

$S = \pi rs$

*Total surface value*

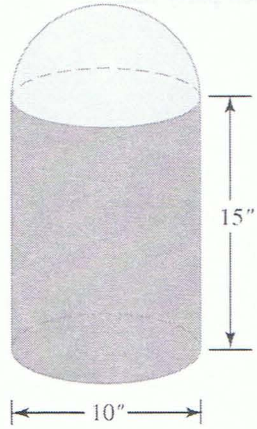
$S = \pi rs + \pi r^2$

*Volume*

$V = \dfrac{1}{3}\pi r^2 h$

$s$ is called the *slant height* of the cone.
$r$ is called the *base radius*.

**EXAMPLE 3** To the nearest tenth, find the volume and surface area for the following figure. The figure is a hemisphere (half a sphere) on top of a right circular cylinder.




**Solution** We see that the diameter of the hemisphere and the cylinder is 10 in., and the height of the cylinder is 15 in.

To find the volume of this solid, we find the volume of each of the hemisphere and the cylinder:

$$V = V_{\text{hemisphere}} + V_{\text{cylinder}}$$ *Since a hemisphere is half a sphere,*

$$V = \frac{1}{2}V_{\text{sphere}} + V_{\text{cylinder}}$$ *Use the formula to get*

$$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + \pi r^2 h$$ *The radius of the sphere and the cylinder is 5 in. (half the diameter); the height of the cylinder is 15 in.; hence, r = 5 and h = 15.*

$$V = \frac{1}{2}\left(\frac{4}{3}\pi(5)^3\right) + \pi(5)^2(15)$$

$$V = \frac{1}{2}\left(\frac{4}{3}\pi(125)\right) + \pi(25)(15)$$ *Simplify.*

$$V = \frac{250\pi}{3} + 375\pi = \frac{250\pi}{3} + \frac{1{,}125\pi}{3} = \frac{1{,}375\pi}{3}$$ *Using a calculator, we get*

$$V = 1{,}439.9$$

Hence, the volume is $\boxed{\text{1,439.9 cubic in.}}$

To find the surface area, we note that we want only the outside portion of the hemisphere, the lateral area of the cylinder, and the bottom base of the cylinder.

$$S = S_{\text{hemisphere}} + S_{\text{lateral area of cylinder}} + S_{\text{bottom base of cylinder}}$$ *Since a hemisphere is half a sphere,*

$$S = \frac{1}{2}S_{\text{sphere}} + S_{\text{lateral area of cylinder}} + S_{\text{bottom base of cylinder}}$$ *Use the formula to get*

$$S = \frac{1}{2}(4\pi r^2) + 2\pi rh + \pi r^2$$ *The radius of the sphere and the cylinder is 5 in.; the height of the cylinder is 15 in.; hence, r = 5 and h = 15.*

$$S = \frac{1}{2}(4\pi(5)^2) + 2\pi(5)(15) + \pi(5)^2$$

$$S = 50\pi + 150\pi + 25\pi$$ *Simplify.*

$$= 225\pi$$

$$S \approx 706.9$$

The surface area is $\boxed{\text{706.9 square in.}}$

## EXAMPLE 4

A party hat is to be made in the shape of a right circular cone, as illustrated in Figure G.76. If the material used to make the hat costs \$0.175 per sq ft, find the material cost of manufacturing one such party hat.

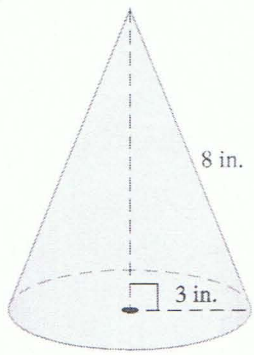



**Figure G.76** The party hat of Example 4

**Solution** The material used in making the party hat goes into creating the lateral surface area of the cone. To compute the cost of one hat, we must compute its lateral surface area in sq ft and then multiply this surface area by the cost per sq ft for the material.

The lateral surface area is $A = \pi rs = \pi(3)(8) \approx 75.4$ sq in. To convert this area into sq ft, we divide by 144, which is the number of sq in. in 1 sq ft. Thus,

$$75.4 \text{ sq in.} = \frac{75.4}{144} \approx 0.52 \text{ sq ft}$$

Finally, to compute the cost of the hat, we multiply this area by the cost of the material per sq ft:

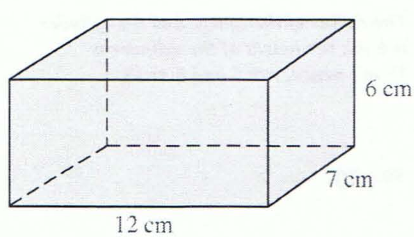
$$\text{Cost of one hat} = 0.52(0.175) \approx 0.09$$

Thus, the cost is about $\boxed{\$0.09 \text{ or } 9 \text{ cents}}$ per hat for the material.

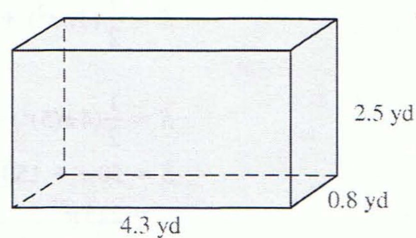
## EXERCISES G.8

*In this exercise set, answers may be left in terms of π unless other instructions are given. In Exercises 1–8, use the given figure to find the total surface area and volume of the solid.*

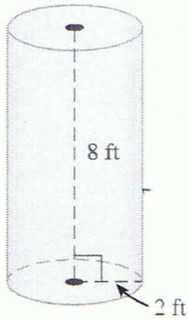
**1.**



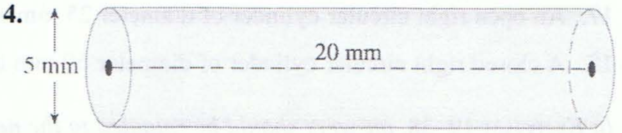

**2.**



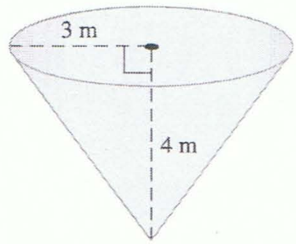
**3.**



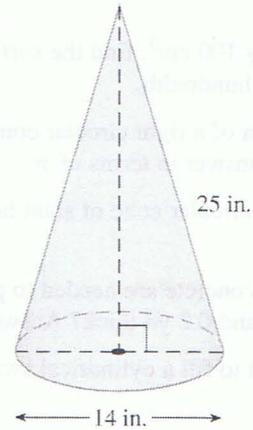

**4.**



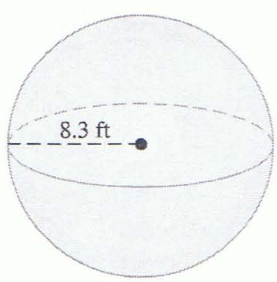

**5.**



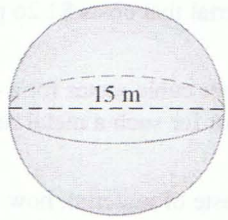

**6.**




**7.**




**8.**




*In Exercises* 9–18, *use the formulas given in this section to compute the total surface area and the volume of the figure described. All answers should be rounded to the nearest tenth.*

**9.** A sphere of radius 3 ft

**10.** A sphere of diameter 3 ft

**11.** A closed right circular cylinder of height 5 cm and radius 4 cm

**12.** A right circular cylinder, closed at only one end, of height 4 cm and radius 5 cm

**13.** A rectangular solid of dimensions 2.4 m by 15 m by 1.8 m

**14.** A cube of side 1.6 mm

**15.** A right circular cone of radius 8 ft and a slant height of 12 ft

**16.** A right circular cone of radius 15.2 cm and a height of 25 cm

**17.** An open right circular cylinder of diameter 25 mm and height 60 mm

**18.** A closed right circular cylinder of diameter 60 mm and height 25 mm

*In Exercises* 19–38, *answers should be rounded to the nearest tenth unless otherwise indicated.*

**19.** A closed right circular cylinder has a radius of 3 m. Find the volume of the cylinder if its lateral surface area is $96\pi$ square meters. Leave your answer in terms of $\pi$.

**20.** A closed right circular cylinder has a radius of 3 m. Find the lateral surface area of the cylinder if its volume is $96\pi$ cubic meters. Leave your answer in terms of $\pi$.

**21.** Find the volume of a sphere if its surface area is $100\pi$ sq cm. Leave your answer in terms of $\pi$.

**22.** If the volume of a sphere is 100 $cm^3$, find the surface area of the sphere. Round your answer to the nearest hundredth.

**23.** Find the lateral surface area of a right circular cone of height 8 in. and volume $96\pi$ cubic in. Leave your answer in terms of $\pi$.

**24.** Find the volume of a right circular cone of slant height 13 mm and lateral surface area 204.2 sq mm.

**25.** How many cubic yards of concrete are needed to pave a driveway that is to be 20 yd long, 3.75 yd wide, and 0.2 yd thick? Answer to the nearest tenth.

**26.** How much water is needed to fill a cylindrical swimming pool of diameter 4 m to a height of 1.4 m?

**27.** A rectangular box is to be used to store wet materials. The inside dimensions of the box are 3.8 ft long, 16 ft wide, and 4.5 ft high. If the inside of the box is to be lined with a waterproof material that costs $1.26 per sq ft, find the cost of lining the box.

**28.** A company charges $12.36 per cubic meter for a certain type of solid cylindrical metal rod. What is the cost for such a metal rod that is 18 meters long and 0.08 meter in diameter?

**29.** Assuming that there is no waste of material, how many solid steel cylinders of diameter 2 in. and height 8 in. can be made from a solid rectangular block of steel that measures 60 in. by 12 in. by 30 in.?

**30.** How many of the steel cylinders described in Exercise 29 can be made from a solid rectangular block that measures 8 ft by 3 ft by 20 ft?

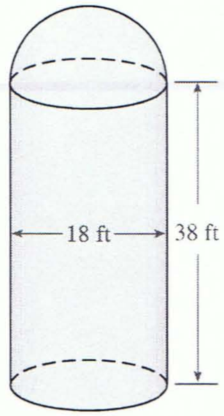
**31.** The cover of a barbecue grill is in the shape of a hemisphere (one half of a sphere) and is made from a material that costs $1.85 per square foot. Find the cost of such a cover if its diameter is 3.5 ft.

**32.** The surface area of a basketball is approximately 285 sq in. Find the radius of the basketball to the nearest tenth of an in..

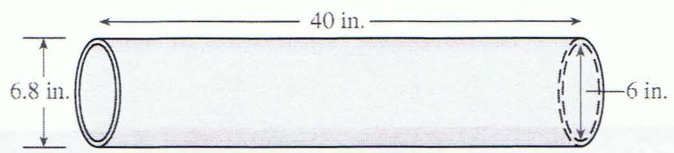
**33.** A conical funnel of radius 15 cm and height 7.5 cm is to be lined with a paper filter. What area of filter paper is needed?

**34.** A hot water heater is in the shape of a right circular cylinder with a radius of 1.2 ft and a height of 5.6 ft. How many square feet of insulation are needed to cover the top and sides of the heater?

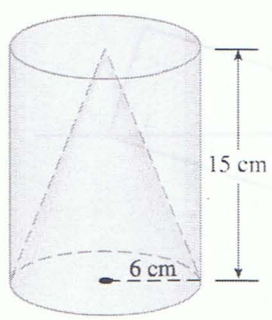
**35.** A silo is built in the form of a cylinder surmounted by a hemisphere, as indicated in the figure. Find the number of cubic feet of grain this silo can hold.



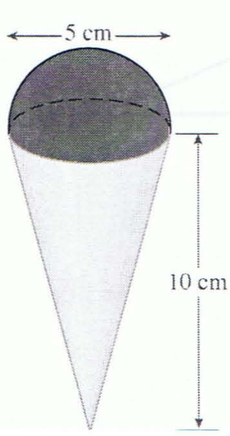

**36.** A cylindrical tube of length 40 in. and diameter 6 in. is surrounded by another tube of diameter 6.8 in., as illustrated in the accompanying figure. The space between the two tubes is used to hold a coolant for the contents of the inner tube. How much coolant can the space between the two tubes hold?




**37.** The accompanying figure illustrates a right circular cylinder with a right circular cone inside it. The radius of both the cylinder and the cone is 6 cm, and the height of both is 15 cm. Find the volume contained in the space outside the cone and inside the cylinder. Leave your answer in terms of $\pi$.




**38.** An "ice cream cone" is formed by placing a hemisphere of diameter 5 cm on top of a right circular cone of height 10 cm, as illustrated in the accompanying figure.



**(a)** If this ice cream cone is completely filled with ice cream, how much ice cream will it contain?

**(b)** If 1 ounce of ice cream has a volume of approximately 29.6 cubic centimeters, how many ounces of ice cream will this cone contain?

## ? Questions for Thought

**39.** Consider a cube of side $x$.

**(a)** Show that the surface area of a cube of side $x$ is $S = 6x^2$.

**(b)** If the edge of a cube is doubled in length, what happens to the surface area? To the volume? [*Hint:* Consider the ratio of the original surface area to the new surface area, and similarly for the volumes.]

**(c)** If the edge of a cube is tripled in length, what happens to the surface area? To the volume? [*Hint:* Consider the ratio of the original surface area to the new surface area, and similarly for the volumes.]

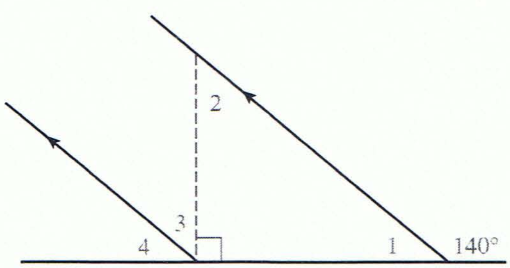
**(d)** Can you generalize the results of parts **(b)** and **(c)** to describe what happens to the surface area and volume of a cube if the length of its edge is multiplied by $k$?

**40.** Describe the similarities in the process of computing area as compared with the process of computing volume.

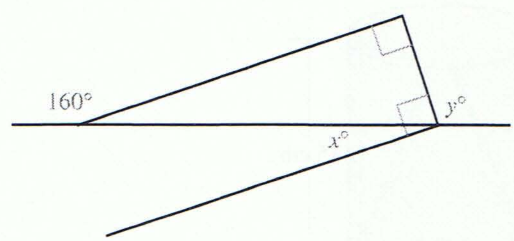
# CHAPTER G REVIEW EXERCISES

*Throughout the following set of review exercises, round your answer to the nearest tenth where necessary.*

**1.** Find the measures of $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$.

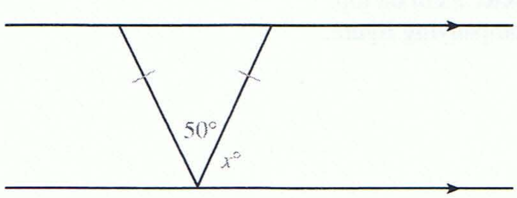



**2.** Find $x$ and $y$.

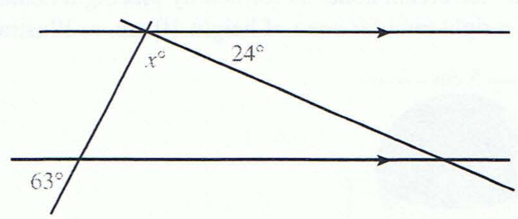



*In Exercises* 3–8, *find x.*

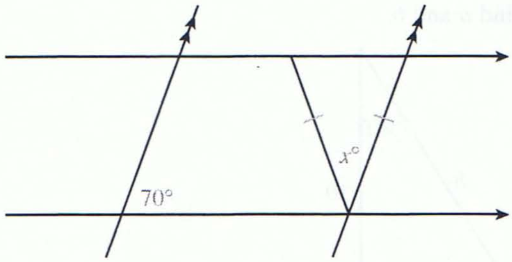
**3.**



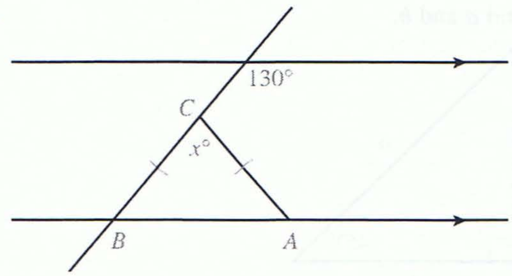

**4.**



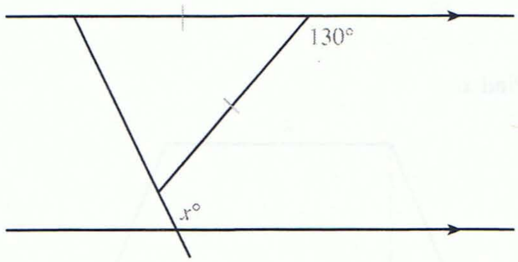
**5.**



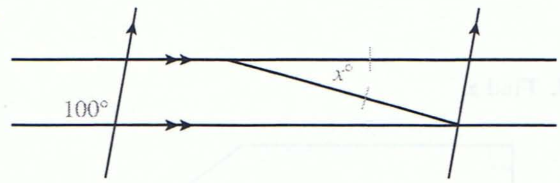

**6.**




**7.**

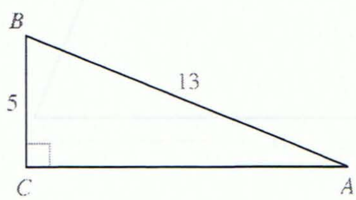



**8.**

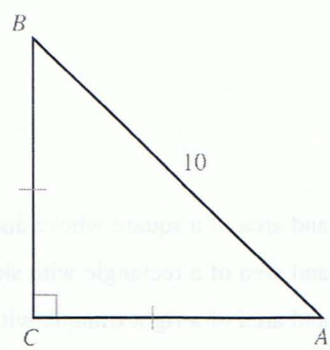



*In Exercises 9–12, find the missing sides of the triangle(s).*

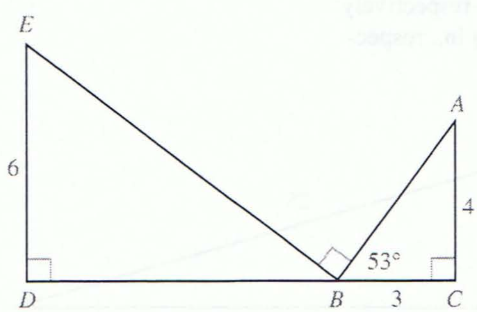
**9.**



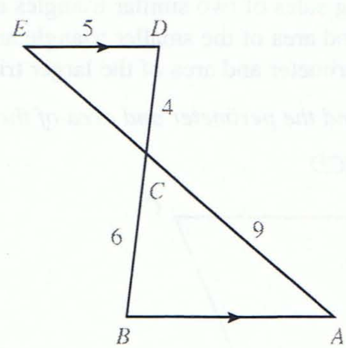

**10.**



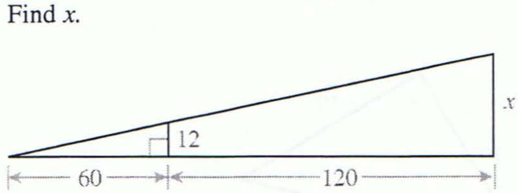

**11.**



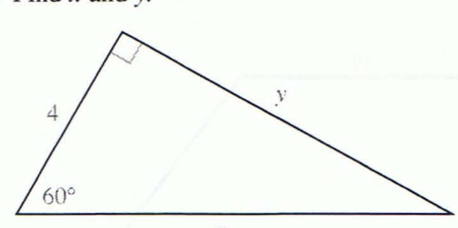

**12.**



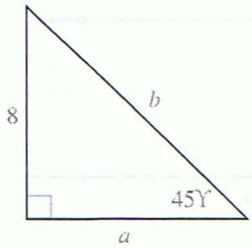

**13.** Find $x$.



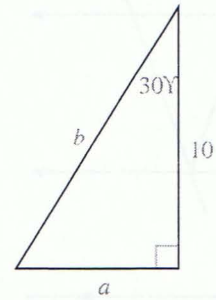

**14.** Find $x$ and $y$.



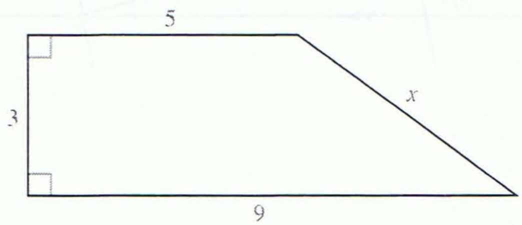
**15.** Find $a$ and $b$.



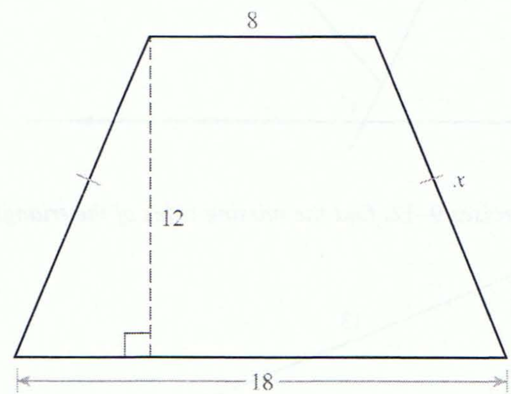

**16.** Find $a$ and $b$.




**17.** Find $x$.




**18.** Find $x$.




**19.** Find the perimeter and area of a square whose diagonal is 12 in.

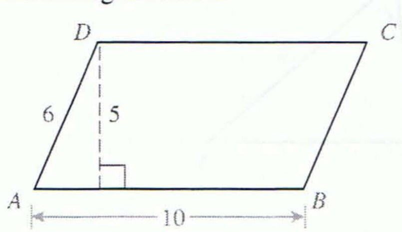
**20.** Find the perimeter and area of a rectangle with side 8 cm and diagonal 10 cm.

**21.** Find the perimeter and area of a right triangle with one leg 5 ft and hypotenuse 9 ft.

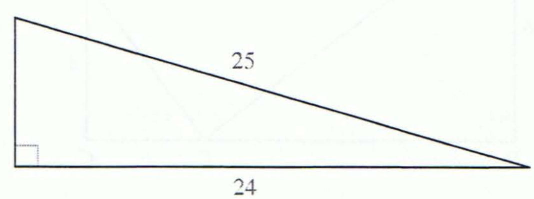
**22.** The corresponding sides of two similar triangles are 3 in. and 2 in., respectively. If the perimeter and area of the smaller triangle are 20 in. and 24 sq in., respectively, find the perimeter and area of the larger triangle.

*In Exercises 23–27, find the perimeter and area of the given figure.*

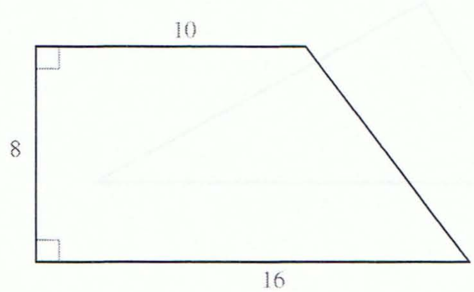
**23.** Parallelogram $ABCD$



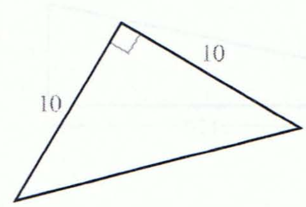

**24.**



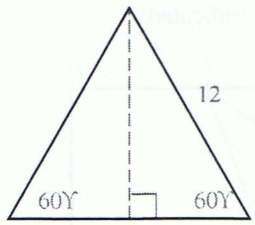

**25.**



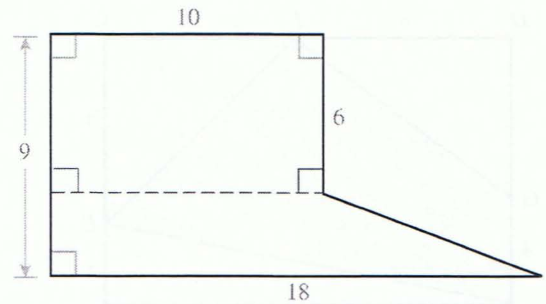

**26.**



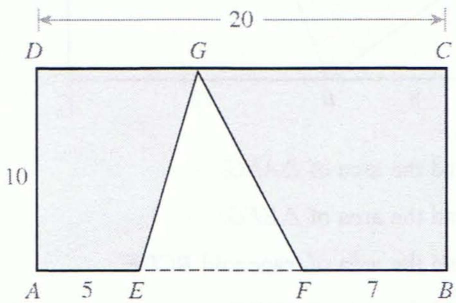
**27.**



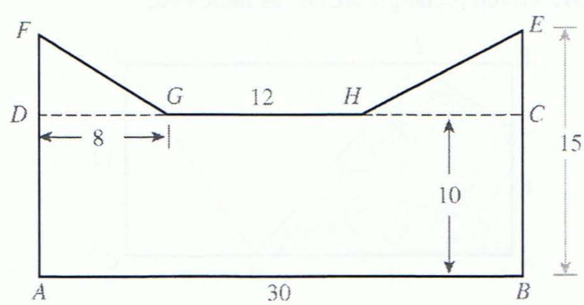

**28.** Find the area of the following figure.



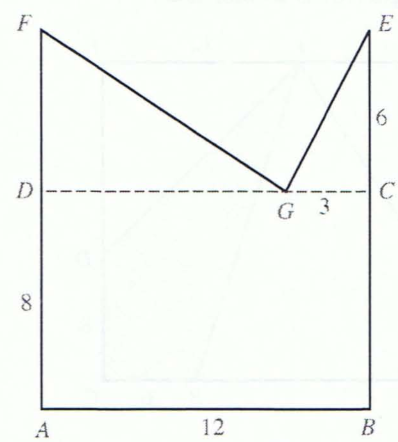

**29.** Find the area of the shaded region. *ABCD* is a rectangle.



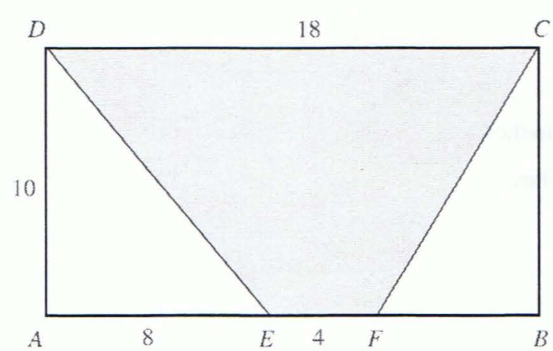

**30.** Find the area of the following figure. *ABCD* is a rectangle and $|\overline{AF}| = |\overline{BE}|$.




**31.** Find the area of the following figure. *ABCD* is a rectangle, and $|\overline{AF}| = |\overline{BE}|$.




**32.** Given rectangle *ABCD* as indicated:



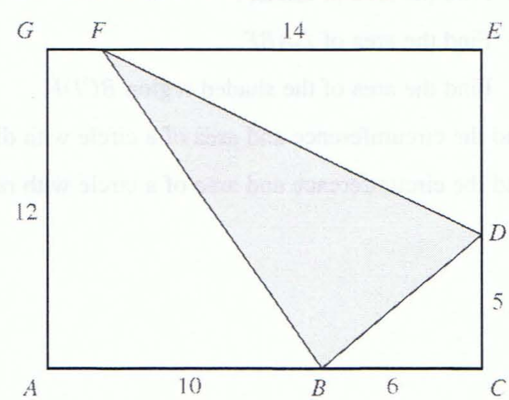

**(a)** Find the area of $\triangle ADE$.

**(b)** Find the area of $\triangle BCF$.

**(c)** What kind of figure is *CDEF*?

**(d)** Find the area of *CDEF*.

**33.** Given rectangle *ACEG* as indicated:



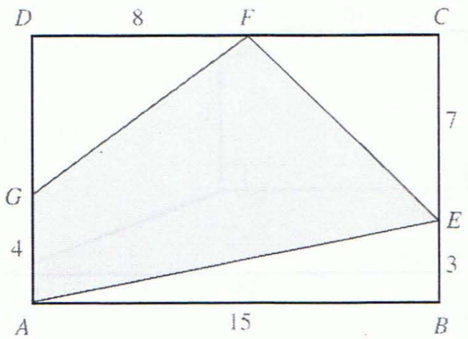

**(a)** Find the area of $\triangle BCD$.

**(b)** Find the area of $\triangle DEF$.

**(c)** Find the area of trapezoid *ABFG*.

**(d)** Find the area of the shaded region.

**34.** Given rectangle *ABCD* as indicated:




**(a)** Find the area of $\triangle ABE$.

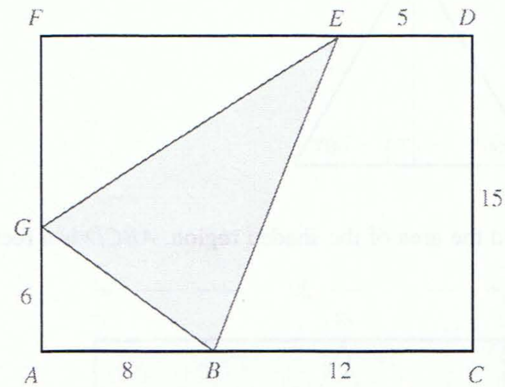
**(b)** Find the area of $\triangle ECF$.

**(c)** Find the area of $\triangle FDG$.

**(d)** Find the area of *ABCD*.

**(e)** Find the area of the shaded region *AEFG*.

**35.** Given rectangle *ACDF* as indicated:




**(a)** Find the area of $\triangle ABG$.

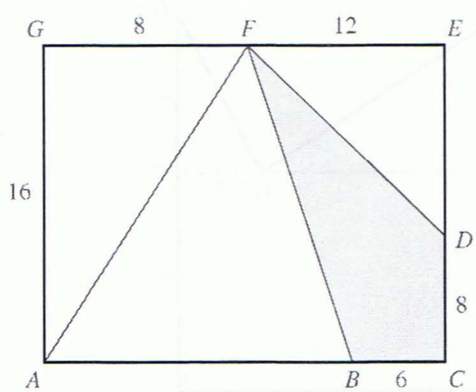
**(b)** Find the area of $\triangle EFG$.

**(c)** Find the area of trapezoid *BCDE*.

**(d)** Find the area of *ACDF*.

**(e)** Find the area of the shaded region.

**36.** Given rectangle *ACEG* as indicated:



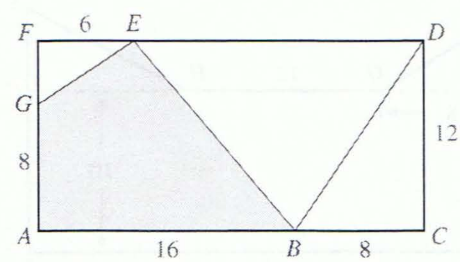

**(a)** Find the area of $\triangle BCD$.

**(b)** Find the area of $\triangle DEF$.

**(c)** Find the area of $\triangle ABF$.

**(d)** Find the area of the shaded region *BCDF*.

**37.** Given rectangle *ACDF* as indicated:




**(a)** Find the area of $\triangle BCD$.

**(b)** Find the area of $\triangle BDE$.

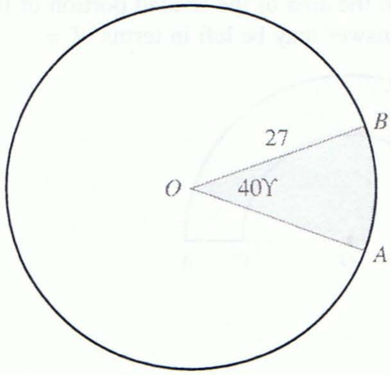
**(c)** Find the area of $\triangle EFG$.

**(d)** Find the area of the shaded region *ABEG*.

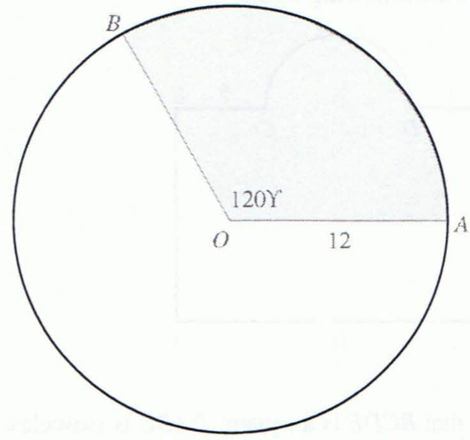
**38.** Find the circumference and area of a circle with diameter 20 inches.

**39.** Find the circumference and area of a circle with radius 16 inches.

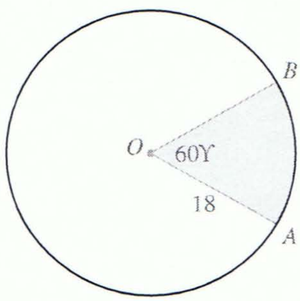
**40.** Find the length of $\widehat{AB}$.



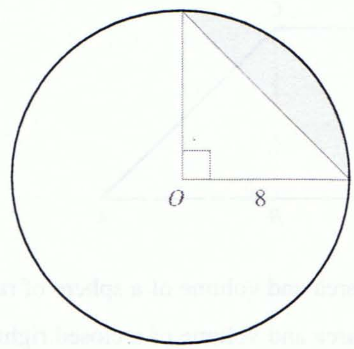

**41.** Find the area of the shaded sector.



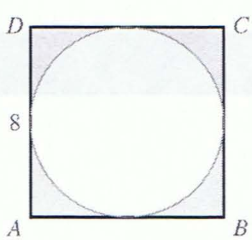

**42.** Find the perimeter of sector $AOB$.



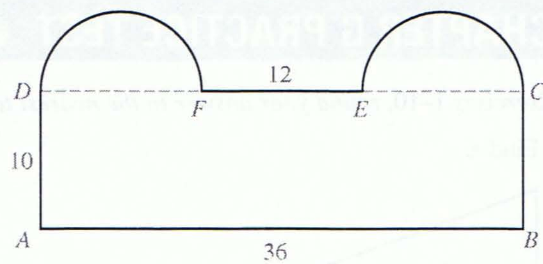

**43.** Find the area of the shaded region.



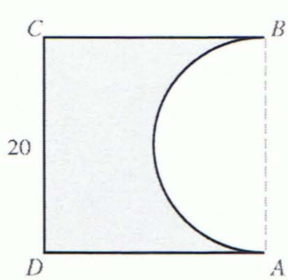

**44.** $ABCD$ is a square. Find the area of the shaded portion of the following figure.



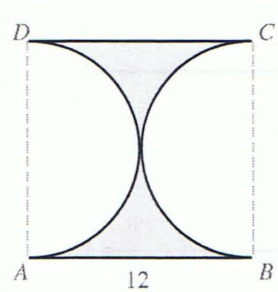

**45.** Find the perimeter of the following figure. $ABCD$ is a rectangle. Arcs $\widehat{CE}$ and $\widehat{DF}$ are congruent semicircles.



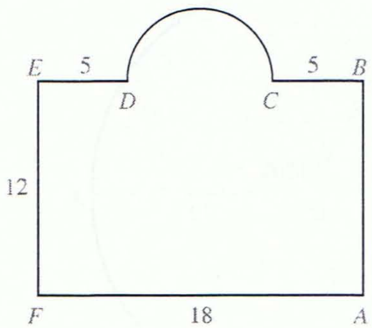

**46.** Find the area and perimeter of the shaded region. $ABCD$ is a square and $\widehat{AB}$ is a semicircle.



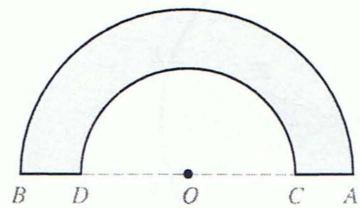

**47.** $ABCD$ is a square of side 12 in. The two arcs are congruent semicircles. Find the area of the shaded region. The answer may be left in terms of $\pi$.



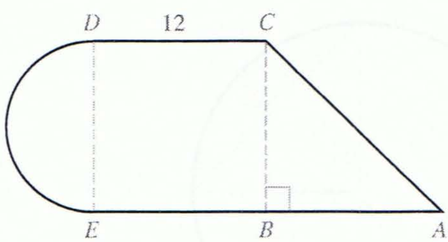
**48.** $ABEF$ is a rectangle and $\widehat{CD}$ is a semicircle. Find the area of the following figure.




**49.** $\widehat{AB}$ and $\widehat{CD}$ are concentric semicircles. $\overline{OA} = 6$ and $\overline{OC} = 4$. Find the area of the shaded portion of the figure. The answer may be left in terms of $\pi$.




**50.** Given that $BCDE$ is a square, $\triangle ABC$ is isosceles, and $\widehat{DE}$ is a semicircle, find the area of the following figure.




**51.** Find the surface area and volume of a rectangular solid whose edges are 3, 5, and 8 cm.

**52.** Find the surface area and volume of a sphere of radius 8 inches.

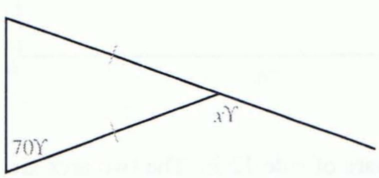
**53.** Find the surface area and volume of a closed right circular cylinder of radius 3 ft and height 10 ft.

**54.** Find the total surface area and volume of a closed right circular cone of base radius 3 m and height 4 m.

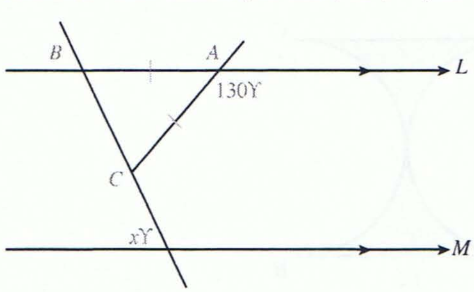
## CHAPTER G PRACTICE TEST

*In Exercises* 1–10, *round your answer to the nearest tenth where necessary.*

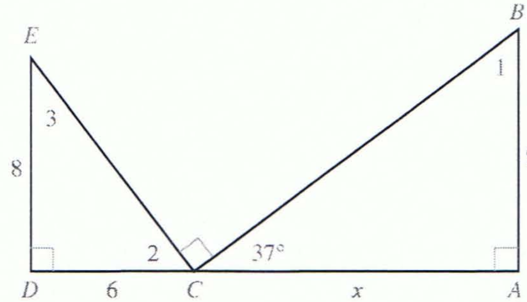
**1.** Find $x$.



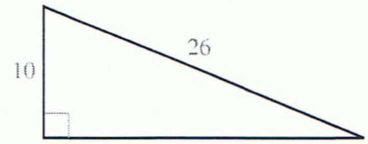

**2.** Find the value of $x$. $L \parallel M$ and $|\overline{AB}| = |\overline{AC}|$.



**3.** Use the accompanying figure to find $\angle 1$, $\angle 2$, $\angle 3$, and the length of line segment $\overline{AC}$ labeled as $x$.



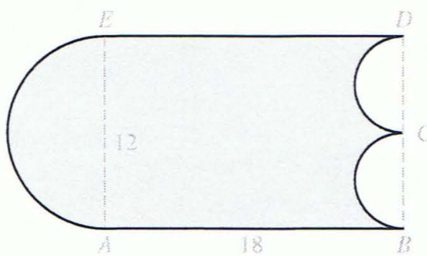

**4.** Find the perimeter and area of the following triangle.



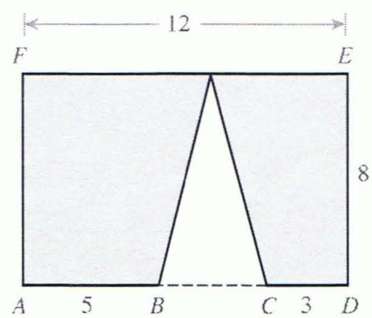

**5.** Find the area of an equilateral triangle of side 6 in.

**6.** Find the perimeter of a 60° sector of a circle with radius 12 cm.

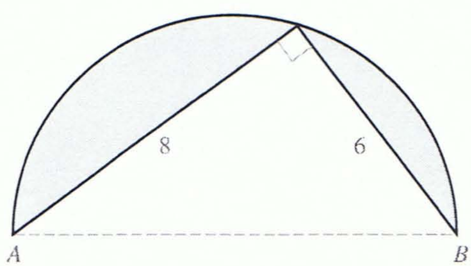
**7.** Given that $ABDE$ is a rectangle, arc $\widehat{AE}$ is a semicircle, and arcs $\widehat{BC}$ and $\widehat{CD}$ are congruent semicircles, find the area of the figure.




**8.** Find the area of the shaded portion of the following figure. $ADEF$ is a rectangle.




**9.** Find the area of the shaded portion of the following figure. $\widehat{AB}$ is a semicircle.




**10.** Find the cost of constructing the following closed rectangular box from material that costs \$2.50 per square foot.